This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

38.

976.

THE

DOCTRINE OF LIMITS.

16-

THE DOCTRINE OF LIMITS,

WITH ITS APPLICATIONS; NAMELY,

CONIC SECTIONS,
THE FIRST THREE SECTIONS OF NEWTON,
THE DIFFERENTIAL CALCULUS.

A PORTION OF A COURSE OF UNIVERSITY EDUCATION.

BY THE

Rev. WILLIAM WHEWELL, M.A.

FELLOW AND TUTOR OF TRINITY COLLEGE,
AUTHOR OF A HISTORY OF THE INDUCTIVE SCIENCES, &c.

BIBLIOTHECA BODLEIANA

CAMBRIDGE:
PRINTED AT THE UNIVERSITY PRESS, FOR
J. AND J. J. DEIGHTON, CAMBRIDGE;
AND
JOHN W. PARKER, LONDON.

M.DCCC.XXXVIII.

975.

CONTENTS.

BOOK I.

PAGE
APPLICATION OF ALGEBRA TO GEOMETRY 1

BOOK II.

THE DOCTRINE OF LIMITS 18

BOOK III.

CONIC SECTIONS 45

BOOK IV.

GEOMETRICAL SOLUTION OF THE DIRECT PROBLEM OF CENTRAL FORCES 73

BOOK V.

THE DIFFERENTIAL CALCULUS.

Definitions 109
Rules of Differentiation of Algebraical Functions 110
Application of First Differential Coefficients to Curves defined by Rectangular Co-ordinates 116
Rules of Differentiation of Circular Functions 122
Application of First Differential Coefficients to Polar Equations of Curves 126
Application of higher Differentials to Curves 134
Application of Differentials to Expansions, including Differentiation of Exponentials, and resolution of Trinomials 137

BOOK VI.

THE INTEGRAL CALCULUS.

Rules of Integration 151
Integration of Natural Algebraical Fractions 157
Reduction of Binomial Integrals 162
Examples of Integration 167

PREFACE.

THIS Treatise on the Doctrine of Limits is intended to contain all those portions of a mathematical course of reading which it appears desirable to recommend to students, as their employment during the *second year* of their residence in the University; with the exception of the Elements of Mechanics, on which I have published separate works. It will, I think, be found convenient to have this group of subjects thus brought together; because it may not only save the trouble of reference to many books, but may prevent the omissions and repetitions which occur when the system is made up of detached fragments; and may secure that connexion and coherence in the reasoning, which are often wanting when a number of different persons undertake to construct each his section of the logical chain.

But though these considerations appear to render such a work as the present desirable, they would probably not have been sufficient to induce me to encounter the labour and difficulty of writing it, if I had not thought that it would illustrate and promote the genuine objects of the study of Mathematics as part of a University education. I have already endeavoured to define those objects, and to extend our means of attaining them, and I will now make a few further remarks on the subject, with a special reference to the present Treatise.

All exact knowledge supposes the mind to be able to apply, steadily and clearly, not only the processes of reasoning, but also certain fundamental ideas; and it is one main office of a liberal education to fix and develope these ideas. The ideas of Space and of Number are the subject matter of Geometry, of Arithmetic, and of Algebra in its character of Universal Arithmetic: and since all our knowledge, relative to the external world, must be subject to the conditions of space and number, the elementary portions of mathematics just mentioned are, rightly and necessarily, made the basis of all intellectual education. If we advance further in mathematical study, with the view of its thus serving as an intellectual discipline, what other ideas do we thus bring into activity and use? I reply, that the main general ideas which we have next to introduce, and which consequently should be the governing principles of the studies of the second stage of a liberal education, are the following:—the mechanical ideas of *Force* and *Body*, with their various modifications; the idea of the *Symmetry* of symbolical expressions;—the idea of the *Universal Interpretation* of symbols, including as an important branch of this, the *Application of Algebra to Geometry*;—and the idea of a *Limit*. In the present volume, I trust there will be found exemplifications and applications of all these ideas, sufficient to form the basis of a system of study, such as our object requires; although the consequences of the last of them, the Doctrine of Limits, form the leading subject of the book.

It will be easily seen that, according to the course of study hitherto usually prevalent in this University, after the first year's studies are completed, all the ideas just noticed have been brought into use, although they may not always have been contemplated distinctly and separately. Mechanics has usually been a prominent portion of the studies of the second year. Of the ideas which form the groundwork of this science, and of the mode in which they may be evolved into a scientific system, I have elsewhere given my views*. The idea of Symmetry is introduced when we consider the Symmetrical Functions of the Roots of Equations, and is thus the basis of the whole Theory of Equations: it is moreover of extensive use and influence in the application of Algebra to Geometry. The Rules of Interpretation of Symbols are well exemplified by the investigations in which co-ordinates are used; and the principles of such rules, in their most general form, have of late years been exhibited in an able and striking manner†. The Doctrine of Limits has usually been studied in its principles in the First Section of Newton's Principia; and in its applications in the Second and Third Sections, and in the sequel to the student's progress, in the remainder of that immortal work. The idea of a Limit is also, as I have elsewhere shewn‡, necessarily involved in every application of the Differential Calculus, whether to curves or to mechanical problems. And thus our course of study in the second year, in this University, has really been an intellectual discipline of a high order; exercising the student's mind to a definite apprehension and ready use of the abstruse and comprehensive ideas of which we have spoken; and opening to him the large and rich storehouses of knowledge, of which these ideas supply the key.

In the following work I have somewhat extended the usual domain of the Doctrine of Limits, by including in it all those portions of Conic Sections which depend upon Tangents, and consequently all which concerns Conjugate

---

* See The Mechanical Euclid.

† See Professor Peacock's Algebra.

‡ On the Study of Mathematics, &c.

Diameters. It appears to me that no other satisfactory basis can be found for any part of our knowledge of curves (except so far as our propositions only refer to straight lines intercepted by the curves). The Axiom of Archimedes, as it is called, respecting the length of arcs of curves, cannot be rendered entirely satisfactory in any other manner, as I have elsewhere urged*. And although the usual negative Definition of a Tangent may be made the foundation of the properties of some figures by reasoning *ex absurdo;* the direct proof which offers itself when we define the tangent as the Limit of a line cutting in two points, resulting when the two become one, appears to be a more genuine demonstration.

With this view I found it necessary to separate the properties of Conic Sections into two parts; the first part (Book I.) being independent of the idea of a limit, (which forms the subject of Book II.) and, besides its being necessary to the proof of the following portions of my subject, serving to illustrate the ideas of Symbolical Symmetry, and Interpretation, and the use of Co-ordinates. The other part of the properties of Conic Sections (Book III.) comes after the Doctrine of Limits, on which it depends; and besides its value in itself, supplies those propositions which are used in treating of Central Forces. (Book IV.)

It thus happens, that my aim in writing a Treatise on Conic Sections has been different from that of most of those mathematicians who have preceded me in such an attempt. Their object has in general been to render their work as homogeneous and symmetrical as possible. They have endeavoured to produce a system which should be rigorously geometrical, with no taint of symbolical reasoning; or else one which should be purely analytical: and they have sought

---

* On the Study of Mathematics.

to make the demonstrations for the ellipse and hyperbola correspond as exactly as possible. My aim has been to employ such demonstrations as should illustrate some principle or some process of extensive use; and hence I have sought variety rather than uniformity in the kind of proof. I have demonstrated algebraically the properties of Conic Sections, which do not depend upon tangents; while I have given geometrical proofs of several other properties. I have deduced the property of the rectangles of oblique co-ordinates in the ellipse from the consideration of the *circular projection* of the ellipse;—a mode of considering the ellipse, which it is very useful for the student to be acquainted with: while in the hyperbola I have deduced the same proposition from the general proportionality of the segments of parallel lines; —a property which is true of all the Conic Sections. Thus the equality of all the tangential parallelograms, in the ellipse and in the hyperbola, is connected with two quite different sets of properties in the two curves; although this identity of property results from a clear analogy of definition, which connection, by adopting another line of reasoning, might be brought into view. By the method which I have adopted, it appears to me that the subject of Conic Sections is rendered more compendious than it can otherwise be made.

In Book IV, which contains the application of the Doctrine of Limits to the direct Problem of Central Forces, I have followed almost exactly the steps of Newton in the second and third Sections of the Principia. It appeared to me that little could be gained by any deviation, in the way of evidence or convenience: and the interest which belongs to these problems as features in the history of the greatest scientific discovery ever made, renders it desirable that the student should see the solutions in the form in which they came from the hand of the great master. I have only modified them so far as to obtain an equation instead of a proportion for the value of the force. I have, however, added to Newton's propositions a few others, which it is inconvenient to the student not to know.

The Differential Calculus has, from the time of Dalembert, most commonly been treated as an application of the Idea of a Limit, as I have treated it in Book v. The attempt made by Lagrange to evade the use of this idea, however ingenious, is quite incapable of being realized, if this Calculus is to be employed in the solution of the problems which give it its meaning and value. The temporary favour which the project found in the eyes of some mathematicians, arose, as I conceive, from the persuasion that mathematical truths are exhibited in their most genuine shape when they are made to depend upon definitions alone; an opinion of which I hope I have made the falsity apparent*. I have endeavoured to avoid the incoherencies and defects which are often introduced in the attempts to develope the fundamental idea into a systematic Calculus.

I have added a book (Book vi.) on the Integral Calculus, which may conveniently be studied to a certain extent in the student's second year of residence. I have given certain processes of integration, but principally as examples of the methods which may be used. I have avoided following out this subject in a complete and systematic manner; a task which would have led me into too wide a field; and also, beyond the stage to which I wish to conduct my readers.

I have not attempted to avoid coincidences with other books in the processes and proofs which I have given. We who at this time of day write on subjects so often treated of before, are all too dependent on our predecessors, whether we know it or not, to have much right to complain of those who borrow from us. In further explanation of such coincidences, I may observe, that a few peculiar points which may have appeared in print before, as the proof of the properties of the ellipse by means of the circular projection, and the Rules for the Reduction of Integrals (Book VI. Art. 31.) were originally, or at least independently, devised by me, and communicated to friends many years ago.

---

* See Mechanical Euclid.

If I shall have succeeded in producing a manual which brings together the mathematical studies of the second year in a convenient and satisfactory form, and in such a manner as to facilitate their operation in the way of intellectual discipline, I shall think my labour well bestowed. It is not likely that I shall hereafter endeavour to carry on to an ulterior point the scheme of mathematical education of which this volume and the Mechanical Euclid may be considered as portions. But I may take the liberty of stating here, in what direction I should prosecute my labours, if I should ever be led to make the attempt. My aim would still be to furnish the student's mind with some additional wealth, from among those comprehensive and fundamental Ideas on which science depends. I should wish, for example, to enable our young men to master two ideas of great importance and interest:—that of a *Wave* and that of *Polarity*. The former of these offers itself in the course of the sciences of Hydronamics and Optics:—the latter occurs in various forms as the basis of a large portion of Optics, Magnetism, Electricity of all kinds, and Chemistry. My object would be, not to use any complex mathematical processes in tracing the consequences of these ideas; but to apply them in such cases, as without much mathematical reasoning might tend to make them fixed and clear in the reader's mind. It would be easy to shew how far this has been from a common case. The general notion of a Wave, as a transfer of *form* distinct from a transfer of *matter*, has been a stumbling-block to most unmathematical reasoners; and the confusion and perplexity which have arisen from the attempts to grapple with this notion, have appeared on various occasions. Now this notion is the basis of the Undulatory Theory of Light; as well as of all reasonings concerning the waves of water; and it is capable of being illustrated in a very elementary manner, to those who will give the subject a steady attention. Again, the notion of Polarity has been so far obscure among men in general, that many, and even mathematicians of note, have asserted that the phrase "Polarization of Light," was an arbitrary and unmeaning term. Yet it will be seen by a little consideration, that this is the only just phrase for the case; since Polarity, or Polarization, is the universal mode of describing *an opposition of properties depending upon an opposition of positions*. The same idea, variously modified, might be illustrated by the fundamental doctrines of Magnetism, Galvanism, and the like; and when the student has gone through such elementary studies, he would be far better prepared, than without them he can be, to derive intellectual profit from the beautiful analysis which is applicable to such cases [*].

By extending our mathematical studies in this manner, always insisting upon the most rigorous synthetical exposition of the elementary principles of each added portion, our mathematical education might become an intellectual discipline, such as the world has never yet seen, and such as it has not even learnt to hope for. Yet such an education alone can provide for the future progress and diffusion of real knowledge, and for the prosperity and advance of that civilisation which depends upon the diffusion of knowledge. It has been, and is, my ambition to contribute my efforts to the object of securing to this University a large share in the merit and glory of preserving, improving, and rendering effectual, an education thus truly liberal.

---

[*] See Mr Airy's Tracts, for the Undulatory Theory; and Mr Murphy on The Principles of Heat and Electricity, for the other subjects.

I will proceed to make a few remarks on the foundations of the Doctrine of Limits.

1. In the beginning of Book II, I have stated the fundamental principles of this doctrine, under the head "Definitions and Axioms." I have already observed, in reference to the Principles of Geometry and Mechanics, that there is no essential distinction between Definitions and Axioms. No principles of either of these kinds, it has been shown*, can be the basis of truth, except the reasoner possesses in his mind the fundamental idea which they involve; and when the idea is there, the various modes of apprehending it which the reasoning requires, may be expressed either by means of Definitions or Axioms, or both. The same may be said with regard to the Doctrine of Limits. The fundamental idea of a Limit is necessary in the first place; and this idea being clearly entertained, the resulting principles will be manifest, in whatever form they are enunciated. We may, for instance, either *define* the equiangular spiral to be the Limit of the polygon whose sides cut all the rays at equal angles; or we may define the equiangular spiral to be the curve which cuts all its rays at equal angles, and then state as an *axiom*, that the equiangular spiral is the limit

---

* Remarks on Mathematical Reasoning in the Mechanical Euclid.

of a polygon all whose sides make equal angles with the rays. In either case we have the same principles to refer to in our reasoning; and in either case the foundation of our conviction is our idea of a Limit.

2. No rigorous reasoning concerning the tangent of the equiangular spiral can be carried on, except the idea of a Limit is referred to, either in a Definition or in an Axiom. In a former publication, containing the First Three Sections of Newton, I had overlooked this necessity, and had attempted to prove the properties of the equiangular spiral from a Definition not involving the conception of Limits. The reader who examines that demonstration carefully, or any other which is constructed on similar grounds, will find that there is one step in the proof altogether inconclusive. And in like manner we may say in general, that no proposition can contain any assertion respecting a Limit in its conclusion, except the idea of a Limit be involved either in the Definitions or in the Axioms which are its foundation. Any pretence to establish such an assertion under other conditions will involve some fallacy or other.

3. The peculiar form of the fallacy, as it occurs in the demonstration formerly given respecting the equiangular spiral, was one of which the inconclusive character is manifest enough, although it may often be overlooked. In order to prove that the ratio of two rays R, S is the same as the ratio of two other rays r, s, a number of intermediate rays were interposed in each case, as R', R'', R''' &c., and r', r'', r''' &c. It was then shown that the ratios of R to R' and r to r' are *ultimately* equal, as also of R' to R'' and r' to r'', and so on; and hence, assuming that all these ratios might be compounded, it followed that the ratios of R to S and of r to s are equal. But it is manifest that this reasoning is fallacious. For if it were

allowable, we might in the same manner argue, that all the ratios of R to R', R' to R'', R'' to R''', and so on as far as S, are ultimately ratios of equality, when the number of intermediate magnitudes R', R'', R''' &c. becomes indefinitely great: and therefore, compounding these ratios, the ratio of R to S is a ratio of equality, whatever be R and S; which is a manifest absurdity.

4. If we look for the error of this reasoning, we shall perceive it to reside in this; that we have compounded an *indefinite* number of proportions which hold good *ultimately* only, and of which the number becomes indefinitely great by the same supposition by which each proportion becomes ultimately true. We may rightly compound an indefinite number of exact proportions, or a finite number of ultimate proportions: but if the number be indefinite while the proportions are ultimate only, the combination of all these proportions is inadmissible. While we try to approach the truth by infinitely diminishing the intervals, we remove it as far as ever by infinitely increasing their number.

5. The same error occurs in a form slightly different in Prop. IV. of the Book upon Hydrostatics, in the "Mechanical Euclid." It is shewn in that Book, Prop. III, that the pressures on two sides of a particle are equal; but this proposition is in fact only ultimately true; and though the notion of Limits could not properly be introduced into that elementary work, the result of that notion is contained in the 7th Axiom of the Hydrostatics, which states that we may neglect the weight of a single particle: (that this Axiom is more correctly replaced by the principle of Limits, I have observed in the Scholium at the end of Book II. of the present work.) And since Prop. III. of the Hydrostatics is thus true ultimately only, we cannot assume the results of its being ap-

plied an indefinite number of times. But this is done in Prop. IV; for the number of particles in a finite part of any side of a vessel will be infinite, and thus we have the same fallacy as before. This fallacy I shall take the earliest opportunity of correcting.

6. From these examples we see that considerable care is requisite in establishing and applying the principles of the Doctrine of Limits. Athough it may be for the most part allowable to say that propositions which are true of finite quantities, are true of limits, this maxim cannot be universally applied; and a series of propositions, such as those in Book II. of the present volume, is necessary, in order to enable us to reason securely about the cases in which the consideration of Limits occurs. The establishment of such a series of propositions by Newton, (the Lemmas of his First Section,) was a very great improvement of the mode of treating such questions: since it enabled him to avoid the harsh hypotheses and insecure processes of his predecessors; and gave to his method a rigour approaching to that of the Elements of Geometry.

7. I have said a rigour *approaching* only to that of Geometry: and I must further venture to say that I have endeavoured to supply what was wanting in this respect; and to make the Doctrine of Limits irreproachable in form, as it is solid in principle. For this purpose it has been necessary for me to modify in some degree, although very slightly, Newton's way of treating this subject. I shall mention one or two such points.

8. I have introduced and stated all the definitions and axioms which I have employed in my reasoning; whereas Newton has commonly omitted to state such principles, and has in many cases left us uncertain what definitions or axioms

he would have selected, if he had thought it necessary to make this part of his work formally exact. He does not define *similar figures*; nor *continued curvature*; nor *finite curvature*; nor does he give the measure of *velocity*, or of *force*, on which his reasonings on those subjects depend. My business was to supply these defects; not however by attempting to divine what Newton's individual view of these points was; but by considering how I could best render the system coherent and complete.

9. In doing this, I have followed to a considerable extent a maxim which appeared to me to be a convenient one; although it has not, so far as I am aware, been steadily acted upon before. The maxim is this, that definitions and axioms are to be constructed so as to refer to finite rather than evanescent quantities. Thus a curve is the limit of a finite portion of a polygon: thus, again, I have not made the reasonings concerning motion to depend upon the definition that velocity is the limit of the ratio of an *evanescent* space to an *evanescent* time; but upon the axiom, that if a body move for a *finite* time, the velocity increasing by steps instead of increasing continuously, the first kind of motion has the second kind for its limit, when the steps are indefinitely increased in number; and I have treated the propositions concerning force in the same manner. It appears to me that this identity of the Continuous with the Limit of the Discontinuous, both being finite, is more plain and palpable, (as soon as the notion of a limit is attained,) than the relations of evanescent quantities; although I have in other works, employed the other method; which indeed is capable of being managed with perfect rigour.

10. It would appear as if Newton had, in some instances, proceeded upon this maxim; and this would afford an ex-

planation of a step in his reasoning of which it is otherwise not easy to give a satisfactory account. In his fifth Lemma he asserts, without proof, that corresponding portions of similar curve lines are proportional. Now, how would he have proved this? By considering the curves as the limits of polygons? There appears to be no other way; and a corollary to the fourth Lemma appears to countenance such an opinion. But this proof, it would seem, can then only be valid when the ultimate equality of the arc and the chord has been established; which is not proved till the seventh Lemma, and then by means of this very fifth Lemma. Here then we seem to reason in a circle: can we find an exit to it?

11. The mode in which I have arranged the steps of this demonstration appears to me unobjectionable. The ultimate equality of a *finite* portion of a polygon and a curve is a fundamental principle of the Doctrine of Limits. It is also a principle, that similar curves are the limits of similar polygons. Hence finite portions of similar curves are proportional; and this enables us to prove that the *evanescent* chord and the arc are equal; which is a truth less evident than the equality of *finite* portions of the polygon and its limiting curve. And thus my first ten propositions are almost identical with Newton's first ten Lemmas.

12. In the definition of a tangent, I found myself obliged to define by means of an evanescent quantity. I do not conceive any definition but this, or one equivalent to this, can lead to general propositions respecting tangents.

13. As the measures of velocity and force, according to which they are the limiting ratios of the increments of space and time, and of velocity and time, are not given as defi-

nitions, it became necessary to prove them as properties which follow from our principles, which is done accordingly in Props. 10 and 12 of Book II.

14. I have already remarked that certain steps in the elementary reasonings of Mechanics and Hydrostatics, cannot, by any effort of simplification, be established without some assumptions equivalent to the Doctrine of Limits. The assumptions thus made, being without any clear ground, may very naturally be expected to leave some impression of dissatisfaction on the minds of those who have a relish for rigorous demonstration. It is the office of the Doctrine of Limits to remove this dissatisfaction, by referring the assumptions just spoken of to their proper origin. This is done in the Scholium at the end of Book II.

15. The axiom which I have adopted as the basis of the reasonings concerning central forces, is (B. II. Art. 15) that if a force be supposed to increase or decrease by steps, instead of varying continuously, the effect at the Limit, when the steps become infinitely numerous, will be the same as in the case of continuous variation. It is here to be understood, although it is not expressed, that the other conditions are the same, in the two cases of discontinuous and continuous variation; that is, the body either falls from rest in both cases, or has in both the same velocity and direction of its motion when the force begins to act.

16. This axiom is not exactly the one which is wanted, if we follow Newton's reasoning in the proof of the fundamental properties of motion by the influence of a central force (B. IV. Prop. 1, 2). Newton's demonstration would require us to assume this axiom;—that if a central force act impulsively at small intervals of time, and these intervals be indefinitely diminished, the limiting effect is the same

as that of a continuous central force. The difference of the two methods is, that here the force is supposed to *act* only at intervals; in our axiom, the force acts constantly, but its *changes* take place at intervals. It would not be difficult either to accommodate the axiom to the proof or the proof to the axiom, and thus to produce an agreement which at present is not exact. For this purpose, the axiom ought to assert, that if the intensity *and direction* of the force vary at intervals only, the limiting effects, when the intervals vanish, will agree with the effects of a force changing its intensity and direction continuously; and the proof might then proceed by the same steps as Newton's First Proposition.

TRINITY COLLEGE,
Jan. 1, 1838.

ERRATA.

PAGE	LINE	
1	12	*for* of A as was *read* if AM was.
5	6	*for* perameter *read* parameter.
14	2	*in the numerator for* a^2 *put* ab.
16	22	*for* perameter *read* parameter.
21	7	*dele* from rest.
50	...	In the Diagram and in the Note *for* N *put* M.
51	18	*for* CN^2 *read* CE^2.
59	8	*for* QCA' *read* QCQ'.
65	17	*for* diameter *read* radius.

65 after line 18 *insert*,

COR. 2. The arcs in a circle are as the angles at the centre: hence if θ be any angle at the centre, measured by the arc of a circle of which the radius is 1, $r\theta$ is the corresponding arc of the circle of which the radius is r.

COR. 3. Hence if s be the length of the arc, $\theta = \frac{s}{r}$.

67 after line 12 *insert*,

COR. 2. To find the curve surface of a frustum of a cone, let r, s be the radius of the base, and the slant side of the cone cut off, l the slant side of the frustum, and $r+k$ the radius of its base. Then $r : s :: k : l$, whence $rl = sk$. And curve surface of frustum $= \pi\{(r+k)(s+l) - rs\} = \pi\{rl + sk + kl\} = \pi\{2rl + kl\} = 2\pi rl + \pi kl$.

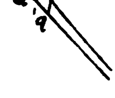
After Prop. 7, Book III, *insert* the following, which is referred to in Book IV, Prop. 18. (The Diagram of this latter Proposition is to be used.)

$$PF \, . \, Px = BC^2.$$

For draw Pm a perpendicular upon AC and produce it to meet CF in x. Also draw Pn perpendicular on CB, and let the tangent PR meet CB in t. Then it may easily be proved that $Cn \, . \, Ct = CB^2$. But Pm is equal to Cn, and Px to Ct, being opposite sides of parallelograms. Therefore $Pm \, . \, Px = CB^2$. Also by similar triangles, $Pm : Px :: PF : Px$; whence $Px \, . \, PF = Pm \, . \, Px$. Hence

$$PF \, . \, Px = BC^2.$$

Also *insert* this Diagram in page 11,



THE

DOCTRINE OF LIMITS,

&c.

BOOK I.

APPLICATION OF ALGEBRA TO GEOMETRY.

1. LET A* be a fixed point in the right line xAX; the position of any point M is estimated by its distance from A. Also if M be supposed to move towards A, then will AM represent successively all quantities less than the first AM: and after M has coincided with A, if it proceed in the same direction to the place M', on the other side of A, then will AM' represent a negative quantity of A as was supposed positive.

If x represent AM, $-x$ will represent AM'.

2. If XAx, YAy be two lines cutting each other at right angles at the point A, the position of any point P, situated in the plane of these lines, is estimated by its distances from the two lines. Thus if PM be drawn perpendicular to XAx, and PN to YAy, the point P is determined, when PN and PM, that is, AM and MP, or AN and NP, are known.

AM is called the *abscissa*, MP the *ordinate*, and AM, MP the *rectangular co-ordinates* of the point P. Also XAx is called the *axis of abscissas*, and YAy the *axis of ordinates*.

3. If YAX be not a right angle, MP is drawn parallel to AY, and AM, MP are *oblique co-ordinates*.

---

* See the Diagram, page 3.

4. What was said in Art. 1, with respect to the signs ($+$ and $-$) of the lines measured on the axis of abscissas, may be applied to those measured on the axis of the ordinates: that is, AN or PM is $+$ or $-$, according as it lies above or below the axis XAx.

Hence if x and y denote the abscissa and ordinate of any point P, the signs of these co-ordinates will be as follows.

In the angle XAY, x is $+$ and y is $+$
YAx, x is $-$ y is $+$
xAy, x is $-$ y is $-$
yAX, x is $+$ y is $-$

Also if θ, $\sin\theta$, $\tan\theta$ represent an angle above AX, and its trigonometrical functions; $-\tan\theta$, $-\sin\theta$, $-\theta$, will represent the same when the angle is below AX.

5. If a line PM move parallel to itself upon the indefinite line xAX, and at the same time increase or decrease in such a manner that some relation between AM and PM constantly remains true; the straight line, or curve traced out by the point P is called the *locus* of P; and the relation between AM and MP, expressed algebraically, is called the *equation to the curve*.

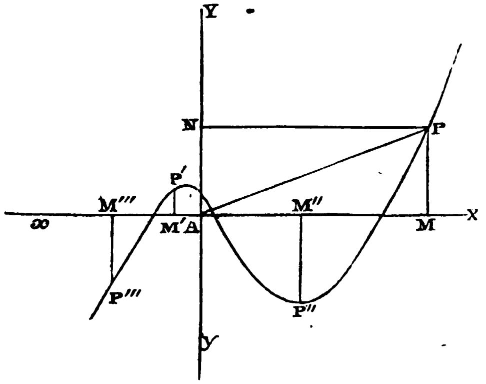
6. If the equation to the curve be given, any number of points in it may be found, and thus its form determined and the curve *traced*.

If the equation between x and y involve numerical coefficients only, take $x = 0, 1, 2, 3$, &c. successively, and determine the corresponding values of y both positive and negative. Again, substitute negative numbers for x, and proceed in the same manner, till a sufficient number of points are determined. The curve must pass through all these points.

EXAMPLE. Let $30y = 4x^3 - 12x^2 - 63x + 30$ be the equation to the curve; we shall find the following results.

When $x = 0$, $y = +1$
$x = 1$, $y = -1.4$
$x = 2$, $y = -3.3$
$x = 3$, $y = -5.3$
$x = 4$, $y = -5.3$
$x = 5$, $y = -2.8$
$x = 6$, $y = +2.8$

Hence the form of the curve is such as is represented



in the figure, to the right of the point *A*. Also to find the remaining part of the curve, take $x = -1, -2, -3$, &c.

When $x = -1$, $y = +2.5$
$x = -2$, $y = +2.3$
$x = -3$, $y = +0.1$
$x = -4$, $y = -5.5$

Hence the curve is such as is represented in the figure.

7. If the equation to the curve involve literal coefficients, the form of the curve may be traced in nearly the same manner.

Example. Let $mx = (x + a)(x - b)(x - c)$ be the equation to the curve. It may be shewn that the form of the curve is nearly the same as before; the axis of x cutting the curve in three points.

8. If the equation to the curve involve only a finite number of algebraic terms, the curve is called *algebraic*; but if the equation cannot be expressed in a finite number of such terms, the curve is called *transcendental.*

9. Lines are divided into *orders* according to their equations; and are said to be of the 1st, 2d, 3d,...n^{th} orders according as their equations are of 1, 2, 3,...n dimensions.

It will be shewn hereafter that an equation of one dimension belongs to a straight line. Hence those curves whose equations are of two dimensions are lines of the second order, but they are the first order of curves; and so on.

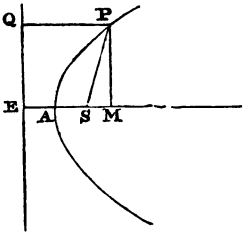
10. A straight line cannot cut a curve of n dimensions (that is a curve whose equation is of n dimensions) in more than n points. (*Wood's* Algebra, Art. 518.)

11. Curves of the 3d, 5th, 7th, &c. orders, and all whose dimensions are odd numbers, must have at least two infinite arcs; whereas the figures of those curves whose dimensions are even, may be like circles or ovals, that are limited within certain limits, beyond which they cannot extend. (*Wood's* Algebra, Art. 519.)

12. If a geometrical property of a line be given, the equation to the line may commonly be found, and the line traced. The following are examples of this.

13. A line is determined by the condition that the distances of any point in the curve from a fixed point, and from a straight line, are equal to each other: to find its equation.

The curve is called a *Parabola.* The fixed point is the *Focus,* the straight line is the *Directrix.*



Let S be the fixed point, EQ the straight line, to which PQ is perpendicular: then $PQ = PS$.

Draw SE perpendicular to EQ, PM perpendicular upon ES. Bisect SE in A, and let $AS = AE = m$, $AM = x$, $MP = y$.

$$PM^2 = SP^2 - SM^2 = PQ^2 - SM^2 = EM^2 - SM^2 = (x+m)^2 - (x-m)^2$$

$$\text{or, } y^2 = 4mx.$$

The quantity $4m$ is the *perameter*. The curve passes through A.

14. A line is determined by the condition that the distances of any point in the curve from two fixed points are together equal to a constant quantity: to find the equation to the curve.

The curve is called an *Ellipse*. The fixed points are the *Foci*.

Let S, H* be the fixed points; PS, PH the distances; therefore $PH + PS$ is constant. Let $PH + PS = 2a$. Draw PM perpendicular to HS; bisect SH in C, and let $CM = x$, $MP = y$. Also let $SC = ae$.

$$\text{Then } HP^2 - HM^2 = PM^2 = SP^2 - SM^2;$$

$$\text{whence } HP^2 - SP^2 = HM^2 - SM^2;$$

$$\text{and } (HP + SP)(HP - SP) = (HM + SM)(HM - SM).$$

$$\text{But } SP + HP = 2a,\ HM + SM = 2ae,\ HM - SM = 2CM = 2x;$$

$$\text{whence } 2a(HP - SP) = 2ae \,.\, 2x,\ HP - SP = 2ex;$$

$$\text{also } HP + SP = 2a;$$

$$\text{therefore } SP = a - ex,\ HP = a + ex.$$

$$\text{Now } PM^2 = SP^2 - SM^2 = (a - ex)^2 - (ae - x)^2$$

$$= a^2(1 - e^2) - x^2(1 - e^2)$$

$$y^2 = (1 - e^2)(a^2 - x^2).$$

15. Let A, a, be the points where CS meets the curve; therefore $SA + HA = 2a$, that is, $2CA = 2a$, and $CA = a$. In like manner $Ca = a$: CA and Ca are each the *semi-axis major*.

---

* See the Diagram, page 6.

Let B be the point where a perpendicular to SH at the point C meets the curve. Therefore $SB = HB$, and $2SB = 2a$, $SB = a$: CB is the *semi-axis minor*.

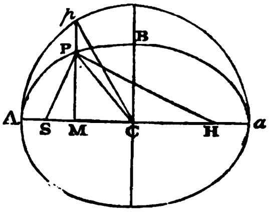
If $CB = b$, $b^2 = SB^2 - SC^2 = a^2 - a^2e^2 = a^2(1 - e^2)$.

16. Hence, $1 - e^2 = \frac{b^2}{a^2}$, and $y^2 = \frac{b^2}{a^2}(a^2 - x^2)$

$$= \frac{y^2}{(a - x)(a + x)} = \frac{b^2}{a^2}, \text{ or } \frac{MP^2}{AM.Ma} = \frac{BC^2}{AC^2}.$$

17. If a circle be described with center C and radius CA, and if MP meet this circle in p, $Mp^2 = Cp^2 - CM^2 = CA^2 - CM^2 = AM.Ma$:

Hence, $\frac{MP^2}{Mp^2} = \frac{b^2}{a^2}$; $MP = \frac{b}{a}Mp$; and the ordinate of the semi-ellipse is in a constant ratio to the ordinate of the semi-circle described on the axis major.

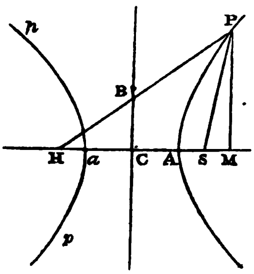


This semi-circle is called the *circular projection* of the semi-ellipse: the point p is the circular projection of the point b, and the line Cp of the line CP.

18. A line is determined by the condition that the distances of any point in the curve from two fixed points shall have their difference equal to a constant quantity: to find its equation.

The curve is called the *Hyperbola*. The fixed points are called the *Foci*.

Let S, H be the fixed points, PS, PH the distances, therefore $PH - PS$ is constant: let $PH - PS = 2a$. Draw PM perpendicular to HS; bisect SH in C; and let $CM = x$, $MP = y$; also let $CS = ae$.



Then $HP^2 - HM^2 = PM^2 = SP^2 - SM^2$:

whence $HP^2 - SP^2 = HM^2 - SM^2$;

and $(HP + SP)(HP - SP) = (HM + SM)(HM - SM)$.

But $HP-SP=2a$, $HM-SM=2ae$, $HM+SM=2CM=2x$;

$$\text{hence } (HP+SP) \,.\, 2a = 2ae \,.\, 2x. \quad HP+SP=2ex.$$

$$\text{Also } HP-SP=2a,$$

$$\text{therefore } HP = ex + a, \quad SP = ex - a.$$

$$\text{Now } PM^2 = SP^2 - SM^2 = (ex-a)^2 - (x-ae)^2,$$

$$y^2 = (e^2-1)(x^2-a^2).$$

The equation gives the curve AP, and also another curve ap, which is the *opposite hyperbola*.

19. Let A, a be the points where CS meets the curve: therefore $SH - SA = 2a$; that is, $2CA = 2a$, and $CA = a$. In like manner $Ca = a$, CA or Ca is the semi-axis major.

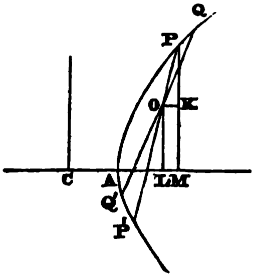
20. Let $\frac{b^2}{a^2} = e^2 - 1$, and $CB = b$,

$$y^2 = \frac{b^2}{a^2}(x^2 - a^2):$$

$$\frac{y^2}{(x-a)(x+a)} = \frac{b^2}{a^2},$$

$$\text{or } \frac{MP^2}{AM \,.\, Ma} = \frac{CB^2}{CA^2}.$$

21. Two lines drawn in a hyperbola cut each other; given the angles at which they are inclined to the axis, to find the ratio of the rectangles of their segments.



Let PP', QQ' be two lines drawn in the hyperbola $PQP'Q'$, which meet in a point O.

Let PM, OL be drawn perpendicular, and OK parallel to the axis.

Let $AL = \alpha$, $LO = \beta$, $AM = x$, $MP = y$; let $OP = r$, and let the angle $POK = \phi$:

$$\text{then } x = AM = AL + OK = \alpha + r\cos\phi,$$

$$y = MP = LO + KP = \beta + r\sin\phi.$$

The equation to the hyperbola is

$$\frac{b^2}{a^2}(x^2 - a^2) = y^2, \text{ or } \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0.$$

Putting for x and y their values, this becomes

$$\left(\frac{\cos^2\phi}{a^2} - \frac{\sin^2\phi}{b^2}\right) r^2 - 2\left(\frac{\alpha\cos\phi}{a^2} - \frac{\beta\sin\phi}{b^2}\right) r + \frac{\alpha^2}{a^2} - \frac{\beta^2}{b^2} - 1 = 0.$$

This is a quadratic equation, in which r will have two values, and these values will be OP, OP'. Let the coefficients of the terms of the equation be represented by m, $2n$, p: then the equation is $mr^2 - 2nr + p = 0$.

Whence the two values of r are

$$\frac{n}{m} + \sqrt{\left(\frac{n^2}{m^2} - \frac{p}{m}\right)},$$

$$\text{and } \frac{n}{m} - \sqrt{\left(\frac{n^2}{m^2} - \frac{p}{m}\right)};$$

and the product of these is $\frac{p}{m}$.

$$\text{Hence the product } OP.OP' \text{ is } \frac{\frac{\alpha^2}{a^2} - \frac{\beta^2}{b^2} - 1}{\frac{\cos^2\phi}{a^2} - \frac{\cos^2\phi}{b^2}}.$$

In like manner if QOQ' make an angle ψ with OK, we shall have

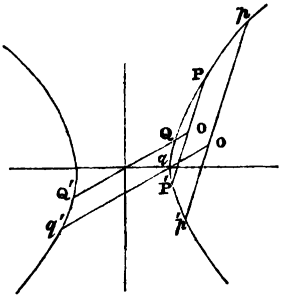
$$OQ \,.\, OQ' = \frac{\frac{\alpha^2}{a^2} - \frac{\beta^2}{b^2} - 1}{\frac{\cos^2\psi}{a^2} - \frac{\sin^2\psi}{b^2}}.$$

Hence $$\frac{OQ \, . \, OQ'}{OP \, . \, OP'} = \frac{\dfrac{\cos^2 \phi}{a^2} - \dfrac{\sin^2 \phi}{b^2}}{\dfrac{\cos^2 \psi}{a^2} - \dfrac{\sin^2 \psi}{b^2}}$$

$$= \frac{b^2 \cos^2 \phi - a^2 \sin^2 \phi}{b^2 \cos^2 \psi - a^2 \sin^2 \psi}.$$

This is true also if the points P' or Q' be in opposite hyperbolas.

22. If in a hyperbola two lines intersect each other, and if two other lines, parallel to the first respectively, also intersect each other; the rectangles of the segments of the first two lines, and the rectangles of the segments of the other two will have the same ratio.



Let PP', QQ' be two lines which intersect in O; pp', qq' two other lines parallel to the former which intersect in o:

then $$OQ \, . \, OQ' : OP \, . \, OP' :: oq \, . \, oq' : op \, . \, op'.$$

For $$\frac{OQ \, . \, OQ'}{OP \, . \, OP'} = \frac{b^2 \cos^2 \phi - a^2 \sin^2 \phi}{b^2 \cos^2 \psi - a^2 \sin^2 \psi} = \frac{oq \, . \, oq'}{op \, . \, op'}.$$

For ϕ is the same for PP' and for pp', because they are parallel: and for the same reason, ψ is the same for QQ' and qq'.

23. A straight line to which a curve approaches perpetually so as to come nearer to it than any finite distance, is called an *asymptote* to the curve.

An asymptote may be found by expanding the equation to the curve according to inverse powers of the abscissa.

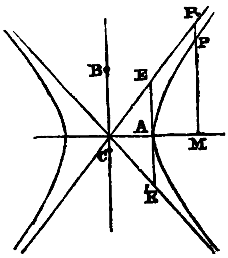
24. To draw an asymptote to the hyperbola.

$$y = \frac{b}{a}\sqrt{(x^2 - a^2)} = \frac{bx}{a}\sqrt{\left(1 - \frac{a^2}{x^2}\right)}$$

$$= \frac{bx}{a}\left(1 - \frac{a^2}{2x^2} - \frac{a^4}{8x^4} + \&c.\right)$$

$$= \frac{bx}{a} - \frac{ab}{2x} - \frac{a^3 b}{8x^3} + \&c.$$

If we draw *AE* perpendicular to *CA* and equal to b, and join *CE*, meeting the ordinate *MP* in *R*, we shall have $MR = \frac{bx}{a}$.



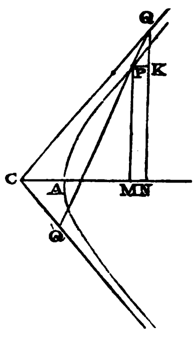
Hence $PR = \frac{ab}{2x} + \frac{a^3 b}{8x^3} - \&c.$

Therefore *PR* becomes indefinitely small, as x or *CM* becomes indefinitely large; and *CR* is an asymptote to the curve *AP*.

There is another asymptote *CE′*, making an equal angle with *AC*.

The asymptotes are also asymptotes of the opposite hyperbola.

25. A straight line being drawn through any point in the curve meeting the two asymptotes of the curve, to find the two portions of this line.



Let *CM*, *MP* be the co-ordinates of *P*, *QN* perpendicular on *CA*, *PK* parallel to *CM*.

Let $CM = x$, $MP = y$, $PQ = r$, angle $QPK = \phi$, angle $QCA = \theta$.

Hence,

$$CN = r\cos\phi + x, \quad NQ = r\sin\phi + y,$$

$$\text{whence, } r\sin\phi + y = \tan\theta\,(r\cos\phi + x)$$

$$y = x\tan\theta - r\,(\sin\phi - \tan\theta\cos\phi).$$

$$\text{But } \frac{b}{a} = \frac{AE}{AC} = \tan\theta, \text{ and } y^2 = \frac{b^2}{a^2}(x^2 - a^2);$$

$$\text{hence, } y^2 = (x^2 - a^2)\tan^2\theta.$$

$$\text{But by the above, } y^2 = x^2\tan^2\theta - 2rx\,(\tan\theta\sin\phi - \tan^2\theta\cos\phi) + r^2(\sin\phi - \tan\theta\cos\phi)^2.$$

$$\text{Hence, } r^2(\sin\phi - \tan\theta\cos\phi)^2 - 2rx\,(\tan\theta\sin\phi - \tan^2\theta\cos\phi) + a^2\tan^2\theta = 0;$$

$$\text{or, } r^2 - \frac{2rx\tan\theta}{\sin\phi - \tan\theta\cos\phi} + \frac{a^2\tan^2\theta}{(\sin\phi - \tan\theta\cos\phi)^2} = 0.$$

This equation gives us for PQ, the value (taking the proper sign)

$$PQ = \frac{x\tan\theta - \sqrt{x^2 - a^2}\tan\theta}{\sin\phi - \tan\theta\cos\phi}.$$

In the same manner we shall find the value of $-PQ'$ if we put $-\tan\theta$ for $\tan\theta$, which gives the line CQ' instead of CQ; that is, from the equation,

$$PQ' = \frac{x\tan\theta + \sqrt{x^2 - a^2}\tan\theta}{\sin\phi + \tan\theta\cos\phi}.$$

Since $\tan^2\theta = \frac{b^2}{a^2}$ we have hence

$$PQ \, . \, PQ' = \frac{a^2\tan^2\theta}{\sin^2\phi - \tan^2\theta\cos^2\phi} = \frac{b^2}{a^2\sin^2\phi - b^2\cos^2\phi}.$$

26. The product of the segments $PQ \, . \, PQ'$ is equal for all parallel lines.

$$\text{For } PQ \, . \, PQ' = \frac{a^2\tan^2\theta}{\sin^2\phi - \tan^2\theta\cos^2\phi} = pq \cdot pq'.$$

Since ϕ is the same when QQ', qq' are parallel.

27. To find the point at which $PQ = PQ'$.

In this case,

$$\frac{x - \sqrt{x^2 - a^2}}{\sin\phi - \tan\theta\cos\phi} = \frac{x + \sqrt{x^2 - a^2}}{\sin\phi + \tan\theta\cos\phi};$$

$$\frac{x - \sqrt{x^2 - a^2}}{x + \sqrt{x^2 - a^2}} = \frac{\sin\phi - \tan\theta\cos\phi}{\sin\phi + \tan\theta\cos\phi}.$$

$$\frac{\sqrt{x^2 - a^2}}{x} = \tan\theta\,\text{cotan}\,\phi;$$

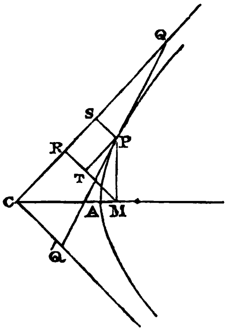
$$x^2 = \frac{a^2}{1 - \tan^2\theta\,.\,\text{cotan}^2\phi}, \quad x = \frac{a}{\sqrt{(1 - \tan^2\theta\,\text{cotan}^2\phi)}}.$$

$$y = \tan\theta\sqrt{x^2 - a^2} = \frac{a\tan^2\theta\,\text{cotan}\,\phi}{\sqrt{(1 - \tan^2\theta\,\text{cotan}^2\phi)}}.$$

Let $PCM = \psi$; $\tan\psi = \frac{y}{x} = \tan^2\theta\,\text{cotan}\,\phi$.

Hence $\tan\phi\tan\psi = \tan^2\theta = \frac{b^2}{a^2}$.

28. At the point where $PQ = PQ'$, to find the product of CQ and CQ'.



Draw MR, PS parallel to $Q'C$; TP parallel to QC. Then it is easily seen that $CM = 2CR\cos\theta$, and $PM = 2PT\sin\theta = 2RS\sin\theta$.

Whence $CR = \frac{x}{2\cos\theta}$, $RS = \frac{y}{2\sin\theta}$;

$$CS = \frac{x}{2\cos\theta} + \frac{y}{2\sin\theta} = \frac{1}{2\cos\theta}\left\{x + \frac{y}{\tan\theta}\right\},$$

And therefore, by last Article,

$$CS = \frac{1}{2\cos\theta}\,\frac{1 + \tan\theta\,\text{cotan}\,\phi}{\sqrt{(1 - \tan^2\theta\,\text{cotan}^2\phi)}}.$$

Hence $CQ = 2CS = \dfrac{a}{\cos\theta}\,\dfrac{1+\tan\theta\cot\phi}{\sqrt{(1-\tan^2\theta\cot^2\phi)}}$.

In like manner, making θ negative, we have

$$CQ' = \frac{a}{\cos\theta}\,\frac{1-\tan\theta\cot\phi}{\sqrt{(1-\tan^2\theta\cot^2\phi)}}.$$

Hence $CQ\,.\,CQ' = \dfrac{a^2}{\cos^2\theta}\,\dfrac{1-\tan^2\theta\cot^2\phi}{1-\tan^2\theta\cot^2\phi}$

$$= \frac{a^2}{\cos^2\theta} = a^2 + a^2\tan^2\theta = a^2 + b^2.$$

29. At the same point, to find CP and PQ.

Let $CP = r$, $PQ = PQ' = s$, $PCM = \psi$. Then $x = r\cos\psi$, $y = r\sin\psi$; and since $y^2 = \tan^2\theta\,(x^2 - a^2)$,

$$r^2\sin^2\psi = \tan^2\theta\,(r^2\cos^2\psi - a^2)$$

$$r = \frac{a\tan\theta}{\sqrt{(\tan^2\theta\cos^2\psi - \sin^2\psi)}} = \frac{ab}{\sqrt{(b^2\cos^2\psi - a^2\sin^2\psi)}}.$$

Also by the triangle CPQ,

$$\frac{s}{r} = \frac{\sin(\theta-\psi)}{\sin(\phi-\theta)} = \frac{\sin\theta\cos\psi - \cos\theta\sin\psi}{\sin\phi\cos\theta - \cos\phi\sin\theta}$$

$$= \frac{\cos\psi}{\cos\phi}\,\frac{\tan\theta - \tan\psi}{\tan\phi - \tan\theta} = \frac{\sin\psi}{\cos\phi}\,.\,\frac{\tan\theta - \tan\psi}{\tan\phi\tan\psi - \tan\theta\tan\psi},$$

or since $\tan\phi\tan\psi = \tan^2\theta$, and r as above;

$$s = \frac{r\sin\psi}{\cos\phi\tan\theta} = \frac{a}{\cos\phi\sqrt{(\tan^2\theta\,.\,\cot^2\psi - 1)}}.$$

But $\tan\theta\cot\psi = \dfrac{\tan\theta}{\tan\psi} = \dfrac{\tan\phi}{\tan\theta} = \cot\theta\tan\phi$.

$$\text{Hence } s = \frac{a}{\cos\phi\sqrt{(\cot^2\theta\tan^2\phi - 1)}}$$

$$= \frac{a^2}{\sqrt{(a^2\sin^2\phi - b^2\cos^2\phi)}}.$$

30. The proofs of the following propositions may be found in *Francœur's* Pure Mathematics, *Waud's* Algebraical Geometry, *Hamilton's* Analytical Geometry, *Lacroix's* Trigonometrie, and other similar works.

The equation to a straight line is $y = ax + b$: where a is the tangent of the angle which the line makes with the axis of x, and b is the distance from the origin at which it intersects the axis of y.

31. Let α, α' be the abscissas, β, β' the ordinates of the points; then the equation to the required line is

$$y - \beta = \frac{\beta' - \beta}{\alpha' - \alpha}(x - \alpha).$$

The two points may be called the points (α, β) and (α', β') respectively.

32. To find the equation to a straight line perpendicular to the line $y = ax + b$, and passing through the point $(\alpha\beta)$.

The equation is $y - \beta = -\dfrac{1}{a}(a - \alpha)$.

33. To find the angle (θ) contained between two given lines $y = ax + b$, $y = a'x + b'$:

$$\text{we have } \tan\theta = \frac{a - a'}{1 + aa'}.$$

If the lines are parallel, $a = a'$; if perpendicular, $a' = -\dfrac{1}{a}$; as in Art. 32.

34. To find the intersection (X, Y) of two lines

$$y = ax + b, \quad y = a'x + b':$$

we have $X = \dfrac{b' - b}{a - a'}$, $\quad Y = \dfrac{ab' - a'b}{a - a'}$.

35. To find the distance between two given points $(\alpha\,\beta)$, $(\alpha'\,\beta')$.

The distance is $\sqrt{\{(\alpha' - \alpha)^2 + (\beta' - \beta)^2\}}$.

36. The equation to a circle of which the radius is γ, and the center $(\alpha\beta)$, is

$$(x - \alpha)^2 + (y - \beta)^2 = \gamma^2.$$

37. Every equation of one dimension in x and y is an equation to a straight line; and the line may be traced by means of the equation.

Examples. To trace the following lines:

(1) $2y + 2x + 4 = 0$.

(2) $y = 2x$.

(3) $y - x + 4 = 0$.

(4) $3y - 2x - 9 = 0$.

(5) $x = a$.

(6) $y = 0$.

38. Every equation of two dimensions in x and y, is an equation to a conic section; either parabola, ellipse, hyperbola, circle, or two straight lines; and the equation being given, we may determine to which conic section the equation belongs, and may trace the curve.

Examples. To determine the conic sections to which the following equations belong:

(1) $y^2 - 2xy + 3x^2 - 2y - 4x + 5 = 0.$

(2) $y^2 - xy + \frac{x^2}{2} - x + \frac{1}{2} = 0.$

(3) $4y^2 + 8xy + 8x^2 + 12x + 8y + 1 = 0.$

(4) $y^2 - 2xy - x^2 - y + 7x - \frac{15}{4} = 0.$

(5) $4y^2 - 8xy + x^2 + 4y + 2x - 2 = 0.$

(6) $y^2 - 2xy + x^2 - 8x + 16 = 0.$

39. A *cone* is a solid generated by the revolution of a right-angled triangle about one of its sides. The angle contained between the fixed side and the hypothenuse of the triangle is the *vertical angle*.

To find the equation of the section of a right cone by a plane.

Let the vertical angle of the cone be β; let the angle which the cutting plane makes with the axis be α, and the distance of this plane from the vertex along the axis $= c$.

If α be greater than β, the section is an ellipse, if α be equal to β, it is a parabola, if α be less than β, it is a hyperbola.

In the ellipse and hyperbola,

$$\text{semi-axis major} = \frac{c \sin \alpha \sin \beta \cos \beta}{\sin (\alpha + \beta) . \sin (\alpha - \beta)}.$$

$$\text{semi-axis minor} = \frac{c \sin^2 \alpha}{\sqrt{\{\sin (\alpha + \beta) . \sin (\alpha - \beta)\}}}.$$

In the parabola the perameter (see Art. 13) is $4c \sin^2 \frac{\beta}{2}$.

40. A conic section may be made to pass through five given points.

41. To *transform* the co-ordinates of a curve; that is, having given the equation to any co-ordinates, to find the equation to any other co-ordinates.

If x, y be the original rectangular co-ordinates, X, Y new oblique co-ordinates, θ the angle which X makes with x, and θ' the angle which Y makes with y; we have

$$x = X\cos\theta + Y\cos\theta',$$

$$y = X\sin\theta + Y\sin\theta'.$$

42. The position of any point may also be determined by *polar co-ordinates*; that is, by knowing the distance AP* (which we may call r), and the angle PAX (which we may call θ): the point A is called the *pole*.

If there be an equation given between r and θ, the locus of P will be a curve determined by this *polar equation*.

43. Given the rectangular equation of a curve, to find the polar equation.

44. If the pole be at the focus, and θ be measured from the axis major, (beginning at the nearest extremity of it) the polar equations to the ellipse and hyperbola, are respectively,

$$r = \frac{a(1-e^2)}{1+e\cos\theta}; \quad r = \frac{a(e^2-1)}{e\cos\theta+1}.$$

45. The polar equation to the parabola is

$$r = \frac{2m}{1+\cos\theta}.$$

---

* See Diagram, p. 3.

BOOK II.

THE DOCTRINE OF LIMITS*.

DEFINITIONS AND AXIOMS.

ART. 1. THE *Limit* of a proposed *magnitude* is that to which the magnitude can approach indefinitely near.

The proposed magnitude is supposed to be given by a hypothetical construction, of which the hypothesis is modified in some indefinitely progressive manner; for instance, some given quantity is supposed to be divided into parts, and the number of these parts is indefinitely increased, and the magnitude of each part indefinitely diminished. A magnitude depending on the given quantity approaches its Limit, as this hypothetical division of the quantity is indefinitely continued; or as the hypothesis approaches its *ultimate form*.

2. The *Limit* of a proposed *Figure* or *Position* is, in like manner, that figure or position to which the given figure or position can approach indefinitely near, by indefinitely extending the hypothesis of the construction.

3. A magnitude is said to be *ultimately equal* to its Limit; and the two are said to be *ultimately in a ratio of equality*.

4. A line or figure *ultimately coincides* with the line or figure which is its Limit.

---

* Newton Principia, Book I. Section I.

5. The Definitions, Axioms, and Propositions in which Limits are concerned, may be enunciated by means of three clauses; of which the first expresses *the hypothetical construction*; the second *its indefinite extension*; the third *the limiting property*.

6. If a finite portion of a curve be taken, and if many points be taken in the curve, so as to form a polygon*, the sides of which are chords, taken in order, of portions of the curve;—

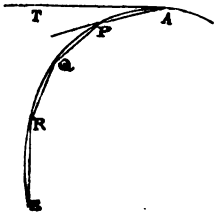
And if the number of these points be increased indefinitely, and the magnitude of each side of the polygon consequently diminished indefinitely;—

The *curve* is the Limit of the polygon.

Let *AB* be a finite portion of a curve, and let points *P, Q, R,* be taken, and chords *AP, PQ, QR, SB* be drawn;—

When the number of these points is indefinitely increased, and the distances, *AP, PQ,* &c. indefinitely diminished;—

The curve *AB* is the limit of the polygon *APQRB*.



7. The construction of Art. 6 being made, and extended indefinitely, and any side of the polygon being produced;—

The limiting position of this line is a *Tangent* to the curve at the point through which it is drawn.

Let *AT* be the limiting position of *AP*, when *AP* is indefinitely diminished; *AT* is a tangent to the curve *AB* at *A*.

8. The construction of Art. 6 being made, and extended indefinitely;—

If the limiting angle made by two successive sides of the polygon (one of them being produced) be not finite at any point, the curve has *continued curvature*.

---

* The term *polygon* is, in this work, employed to describe a broken line made up of a series of straight lines, although the broken line do not return into itself so as to inclose space.

9. Two *curves touch each other* at any point, when they have the same tangent at that point.

10. If a line (BD*) be drawn, making a finite angle with the tangent (AD), this line (BD) is the *subtense of the angle of contact* at the point A; and the intercepted curve (AB) is the *conterminous arc*.

11. If a circle be drawn, touching a curve at any point, and cutting it in a second point;—

And if the second point approach indefinitely to the first;—

The Limiting Circle thus obtained has the *same curvature* as the curve at the point at which they they touch.

If AbB† be any arc of a curve, having AD for its tangent, and if ABb be a circle touching AD and cutting the curve in B; if B approach indefinitely to A, the Limit of the circle ABb has the same curvature as the curve AB at the point A.

12. The *curvature* of a circle is *finite*, when its diameter is finite.

13. If in a finite portion of a curve any polygon be inscribed, and if the figure revolve about any axis, so that the curve may describe a surface of revolution, and the polygon may describe an assemblage of conical surfaces;—

And if the number of sides of the polygon be increased, and the magnitude of each diminished indefinitely;—

The curve surface is the Limit of the assemblage of conical surfaces.

14. When a body moves with a variable finite velocity for a finite time; if this time be divided into many small portions, and if the body be supposed to move during each of these portions in succession with the same velocity, continued constant, which it has at the beginning of that portion;—

---

* See figure to Prop. 7. of this Book.

† See figure to Prop. 9.

And if the number of these portions be increased indefinitely, and the magnitude of each portion consequently diminished indefinitely;—

The motion which takes place on this supposition will have for its Limit the motion produced by the variable velocity.

15. When a body is urged from rest by a variable finite force for any finite time; if this time be divided into many small portions; and if the force be supposed to act upon the body during each of these portions in succession, with the same intensity, continued constant, which it has at the beginning of that portion;—

And if the number of these portions be increased indefinitely, and the magnitude of each portion consequently diminished indefinitely;—

The motion which takes place on this supposition will have for its Limit the motion produced by the variable force.

16. A quantity which is determined by a hypothetical construction, and which, by extending the construction indefinitely, may be made smaller than any finite quantity, *ultimately vanishes.*

17. When a finite magnitude and another finite magnitude have a difference which ultimately vanishes, the former magnitude has the latter for its Limit. See Prop. 1.

18. If A be any finite magnitude, and μ a quantity which ultimately vanishes, μA ultimately vanishes.

19. If a parallelogram have one side which always remains finite, and another which ultimately vanishes, the area of the parallelogram vanishes ultimately.

20. When a figure is determined by a hypothetical construction, the properties which are true when the construction is indefinitely extended, are properties of the limiting figure.

This is expressed by saying, that what is true *up to* the limit is true *at* the limit.

21. *Similar curvilinear figures* are those which are the Limits of similar polygons.

22. If several lines or *rays* be drawn from the same point or *pole*, and if a polygon be constructed, the angles of which are upon the rays, and the sides of which make successively *equal angles* with the successive rays towards the same parts;—

And if the number of rays which fall within a given finite angle be indefinitely increased, and the angles which they make with each other consequently indefinitely diminished;—

The Limit of the polygon is the *equiangular spiral.*

23. If two straight lines be drawn in such a manner that the two extremities of the one coincide ultimately with the two extremities of the other, the two lines are ultimately equal and coincident.

Thus*, if the two straight lines Ab, Ad, have the extremity A common to the two, and if the extremities b and d ultimately coincide, the lines Ab, Ad are ultimately equal and coincident.

24. On the same suppositions, if there be any curve which always lies between the straight lines, and of which the extremities ultimately coincide with those of the straight lines respectively, the curve is ultimately equal to and coincident with either of the straight lines.

Thus*, if the curve Acb be always between the straight lines Ab, Ad, and if its extremity b ultimately coincide with the extremity d, the curve Acb is ultimately equal to and coincident with either of the straight lines Ab and Ad.

25. In reasoning concerning the Limits of magnitudes, we may employ the same axioms, and therefore the same steps and processes, which are true concerning actual quantities.

For example, magnitudes, and the Limits of magnitudes, are either *ultimately* equal, or else ultimately differ by a finite magnitude.

---

* See figure to Prop. 7.

26. If magnitudes are *ultimately* equal to the same, they are ultimately equal to each other.

27. If two magnitudes which are *ultimately* equal be added respectively to two other magnitudes which are ultimately equal, the wholes are respectively ultimately equal.

PROP. I. (NEWTON, Lemma I.)

1. Two quantities which constantly tend towards equality while the hypothesis approaches its ultimate form, and of which the difference, in the course of this approach, becomes less than any finite magnitude, are ultimately equal.

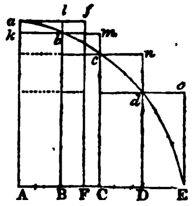
The two quantities must either be ultimately equal, or else ultimately differ by a finite magnitude (Art. 25.). If they are not ultimately equal, let them ultimately have for their difference the finite magnitude D. But by supposition, as the hypothesis approaches its ultimate form, the difference of the two magnitudes becomes less than any finite magnitude, and therefore less than the finite magnitude D. Therefore D is not the ultimate difference of the quantities. Therefore they are not ultimately unequal. Therefore they are ultimately equal.

PROP. II. (NEWTON, Lemma II.)

If in any curvilinear space, bounded by a curve and its two co-ordinates, be inscribed any number of adjacent parallelograms, having their sides parallel to the co-ordinates, and having their bases, along one of the co-ordinates, equal;—

And if the breadths (or equal sides) of these parallelograms be diminished and their number increased indefinitely;—

The sum of the inscribed parallelograms, that of the parallelograms similarly circumscribed, and the curvilinear space itself, will be *ultimately* equal.



Let $AacE$ be the curvilinear space; Ab, Bc, Cd, the inscribed parallelograms; Al, Bm, Cn, Do, the parallelograms similarly circumscribed. The excess of the sum of the latter above the sum of the former, is the sum of the partial excesses, the parallelograms ab, bc, cd, dE; and this is manifestly equal to Al, because the bases bm, cn, do are each equal to AB.

When the breadth AB is diminished indefinitely, and the number of the parallelograms indefinitely increased, the excess Al, of the circumscribed above the inscribed parallelograms, decreases; and the two sums tend constantly to equality. And in approaching the ultimate form of the hypothesis, the difference Al becomes less than any finite magnitude, because the side Aa remains constant and the side AB is indefinitely diminished (Art. 19). Therefore the conditions of Prop. I. are exactly satisfied, and the two quantities, the sums of the inscribed and of the circumscribed parallograms, are *ultimately* equal.

The curvilinear space is always greater than the sum of the inscribed and less than the sum of the circumscribed parallelograms. Therefore the difference between this space and either sum is always less than the difference of the two sums. Therefore the same conditions and reasoning apply to the curvilinear space, as to the two sums of parallelograms, and it is *ultimately* equal to either of the sums.

PROP. III. (NEWTON, Lemma III.)

If the same hypothesis be made, except that the breadths of the parallelograms are not supposed equal;—

And if the breadths be all diminished indefinitely as before;—

The same assertion is true, as in Prop. II.

Let *AF* (fig. p. 24) be the greatest of the breadths of the parallelograms; and let the parallelogram *FAaf* be completed. If the parallelogram *Af* be divided by the lines *mb*, *nc*, *od*, each of the partial excesses *ab*, *bc*, *cd*, *dE*, will be either less than the corresponding portion of the parallelogram *Af*, or equal to it. Hence the whole excess will be less than the parallelogram *Af*.

When all the breadths are diminished indefinitely, the sums of the inscribed and circumscribed parallelograms, and the curvilinear space, tend to equality, as before. And the difference is always less than *Af*, and *Af* *ultimately* becomes less than any finite magnitude. Therefore *ultimately* the inscribed and circumscribed parallelograms and the curvilinear space are all equal.

COR. 1. Every portion of the ultimate sum of the parallelograms is equal to the corresponding portion of the curvilinear space.

COR. 2. The polygonal figure contained by the chords *ab*, *bc*, *cd*, *dE*, is greater than the inscribed and less than the circumscribed series of parallelograms; and hence this polygonal figure is ultimately equal, in all its parts, to the curvilinear figure.

COR. 3. The same may be said of the polygonal figure formed by drawing tangents to the curve at the points *a*, *b*, *c*, *d*, *E*.

COR. 4. If a parallelogram *ABbk* be inscribed in a curve *ABba*, and if the breadth *AB* ultimately vanish, the parallelogram *ABbk* and the portion of the curve *ABba* are ultimately equal.

$$\text{For } \frac{ABba}{ABbk} = \frac{ABbk + akb}{ABbk} = 1 + \frac{akb}{ABbk}.$$

And $\frac{akb}{ABbk}$ is always less than $\frac{albk}{ABbk}$; that is, it is always less than $\frac{lb}{Bb}$; which ultimately vanishes when B comes up to A, since b and l coincide at a.

Therefore $\frac{ABba}{ABbk}$ is ultimately 1.

Cor. 5. In nearly the same manner as in the Proposition, it may be shewn that if, upon the figure AaE (p. 24.), be erected a solid figure; bounded by surfaces all parallel to the same line;—so as to form upon the rectangles Al, &c., parallelopipeds; and upon the curve aE, a figure resembling a *cylinder**;—and if this solid be bounded by another plane parallel to AaE;—the sum of the inscribed parallelopipeds, that of the circumscribed parallelopipeds, and the *cylindroidal* solid, will be *ultimately* equal.

Cor. 6. Also, if the figure AaE revolve about the axis AE, (the ordinates Aa, &c. being perpendicular to AE,) the rectangles Al, &c. will generate cylinders; and the curve aE will generate a figure resembling a portion of a *sphere**; and the sum of the inscribed cylinders, the sum of the circumscribed cylinders, and the *spheroidal* solid, will be *ultimately* equal.

PROP. IV. (NEWTON, Lemma IV.)

If in two figures be inscribed, as in Props. II. and III., two series of parallelograms equal in number;—

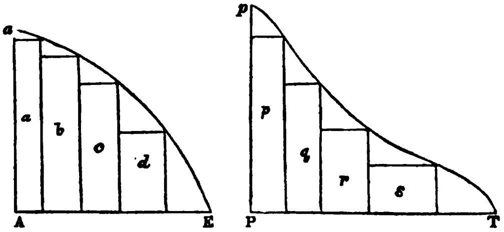
And if, when the breadths of these parallelograms are diminished and their number increased indefinitely, the *ultimate* ratios of the parallelograms in the one figure to those of the other, each to each, be all the same;—

The curvilinear figures are also in the same ratio.

---

* See Scholium of Definitions, Book III.

Let AaE, PpT be the two figures; and let the ratios



$a : p$, $b : q$, $c : r$, $d : s$, &c. of the corresponding parallelograms be *ultimately* all the same.

Let this ratio be $m : n$. And let

$$a : p :: m : n(1+\alpha)$$
$$b : q :: m : n(1+\beta)$$
$$c : r :: m : n(1+\gamma)$$
&c.

where α, β, γ, are quantities which vanish in the ultimate form of the hypothesis.

Hence $na(1+\alpha)=mp$, $nb(1+\beta)=mq$, $nc(1+\gamma)=nr$, &c.; and adding, $n(a+a\alpha+b+b\beta+c+c\gamma+\&c.) = m(p+q+r+\&c.)$

That is, $$\frac{a+b+c+d+a\alpha+b\beta+c\gamma+\&c.}{p+q+r+\&c.}=\frac{m}{n}.$$

That is, $$\frac{a+b+c+\&c.}{p+q+r+\&c.}+\frac{a\alpha+b\beta+c\gamma+\&c.}{p+q+r+\&c.}=\frac{m}{n}.$$

Let μ be the greatest of the quantities α, β, γ, &c.: then, since these all ultimately vanish, μ ultimately vanishes. Now, since α, β, γ, &c. are all less than μ, $a\alpha+b\beta+c\gamma+\&c.$ is less than $a\mu+b\mu+c\mu+\&c.$,

and $\dfrac{a\alpha+b\beta+c\gamma+\&c.}{p+q+r+\&c.}$ is less than $\dfrac{a\mu+b\mu+c\mu+\&c.}{p+q+r+\&c.}$;

that is, than $\mu\dfrac{(a+b+c+\&c.)}{p+q+r+\&c.}$.

But when $\dfrac{a+b+c+\&c.}{p+q+r+\&c.}$ is ultimately finite,

$\mu\dfrac{(a+b+c+\&c.)}{p+q+r+\&c.}$ ultimately vanishes. (Art. 18.)

Therefore *à fortiori*, $\dfrac{a\alpha+b\beta+c\gamma+\&c.}{a+b+c+\&c.}$ ultimately vanishes.

Therefore $\dfrac{a+b+c+\&c.}{p+q+r+\&c.}$ is ultimately equal to $\dfrac{m}{n}$. (Prop. I.)

Therefore ultimately

$$a+b+c+\&c. : p+q+r+\&c. :: m : n. \quad \text{Q. E. D.}$$

COR. Hence if two quantities of any kind be divided into the same number of parts; and those parts, when their number is increased and their magnitude diminished indefinitely, have a given ratio each to each; namely, the first part of the one quantity to the first part of the other; the second to the second, and so on; these two quantities will be in the given ratio.

For the parallelograms in the Prop. being taken to represent the parts, the figures will represent the quantities in question, and their ratio will therefore be the ultimate ratio of the parts.

EUCLID. Book VI. Def. 1.

Similar rectilineal figures are those which have their several angles equal, and the sides above the equal angles proportional.

LEMMA TO PROP. V.

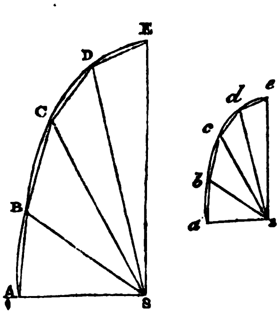
Two similar polygons have corresponding portions of their perimeters proportional to homologous lines in the two polygons; and the areas of the polygons are proportional to the squares of the homologous lines.

Let $SABCDE$, $sabcde$, be two similar polygons. Therefore, as in Euclid, Book VI. Prop. 20, the triangles ABS, abs; BCS, bcs; CDS, cds; DES, des; are similar respectively. Hence

$$BC : BS :: bc : bs,$$

and $$BS : AS :: bs : as;$$

therefore $$BC : AS :: bc : as.$$



In like manner it may be shewn, that

$$CD : AS :: cd : as,$$

and that $$DE : AS :: de : as,$$

also, by similar triangles, $AB : AS :: ab : as$.

Hence (Euclid V. 24. Cor. 2.)

$$AB+BC+CD+DE : AS :: ab+bc+cd+de : as,$$

whence $$AB+BC+CD+DE : ab+bc+cd+de :: AS : as.$$

That is, corresponding portions of the perimeters of the polygons are as their homologous sides.

Also, by Euclid VI. 20, the areas of the polygons are proportional to the squares of the homologous sides.

PROP. V. (NEWTON, Lemma V.)

In two similar curvilinear figures, corresponding finite lines, whether straight lines or portions of the curves, are proportional; and the areas of corresponding parts of the figures are as the squares of homologous lines.

Since (Art. 21) similar curvilinear figures are the limits of similar polygons, let the polygons be drawn of which the two similar curvilinear figures are the limits (see the preceding figures), as $SABCDE$, $sabcde$. Then, by the Lemma, $AB + BC + CD + DE : ab + bc + cd + de :: AS : as$. And this proportion is true when the number of sides of the polygons are increased, and their magnitude diminished indefinitely. Therefore (Art. 20) it is true of the Limits of the perimeters of the polygons *ABCDE*, *abcde*; that is, of the curves *ABCDE*, *abcde*; therefore

curve *ABCDE* : curve *abcde* :: *AS* : *as*.

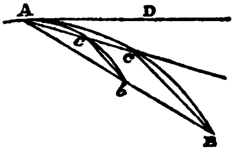
Also, by the Lemma, the polygonal areas

ABCDE : *abcde* :: square of *AS* : square of *as*;

and this is true always, and therefore for the Limit. Therefore curvilinear area *ABCDE* : curvilinear area *abcde* :: square of *AS* : square of *as*.

PROP. VI.

If from a point *A* in a curve *ACB*, any lines *AB*, *AC* be drawn cutting the curve; and if *Ab*, *Ac* be taken always in a constant ratio to *AB*, *AC*, so that a second curve *Acb* may be constructed; the curve *Acb* shall be similar to *ACB*, and the two curves shall have a common tangent at the point *A*.



The triangles *ACB*, *Acb*, produced by this construction, will always be similar; wherefore the angles will be equal and the sides proportional; and therefore the polygons *ACB*, *Acb* will be similar. Hence the curves *ACB*, *Acb*, which are the limits of the polygons, (Art. 21) will be similar curves.

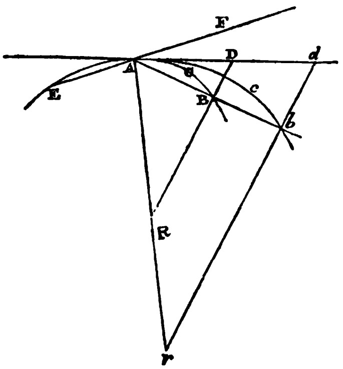
Also the tangent to the curve *ACB* at the point *A* is *AD*, the ultimate position of the side of the polygon *AC*, when the number of sides is increased and the magnitude of each diminished indefinitely (Art. 7); and the tangent of the curve *Acb* is the ultimate position of the line *Ac*, that is, *AC*, under the same circumstances. Therefore the tangent of the two curves is the same line *AD*.

PROP. VII. (NEWTON, Lemma VII.)

If through any point, in a curve of continued curvature, there be drawn a tangent, and also a chord, meeting the curve in a second point: and if the tangent be limited by a *secant* (or line cutting the curve) through the second point;—

And if the second point approach indefinitely near to the first;—

The ultimate ratio of the arc, the chord, and the tangent is a ratio of equality.



Let *AD*, *AB* be the tangent and the chord of *ACB* respectively; *BD* the secant. Let *Ad* be taken always equal to a finite magnitude; let *bd* be drawn parallel to *BD*, meeting *Ab* in *d*; and let a curve *Acb* be constructed as in Prop. 6, which will therefore be similar to *ACB*, and have the same tangent *AD*, by that Proposition. And by Prop. 5, when *AB* is finite, the arc *ACB*, the chord *AB*, and the tangent *AD* will be in the same ratio as the arc *Acb*, the chord *Ab*, and the tangent *Ad*. Therefore, by Art. 20, when *AB* vanishes, the arc *ACB*, the chord *AB*, and the tangent *AD* will still be in the same ratio as the arc *Acb*, the chord *Ab*, and the tangent *Ad*.

But if *EA*, *AB* be two successive sides of the polygon, of which the curve is the limit, if *EA* be produced to *F*, the angle *FAB* ultimately vanishes, because the curve has continued curvature (Art. 8).

And the ultimate position of *EAF* is the tangent *AD* (Art. 7). Therefore the angle *DAB* ultimately vanishes.

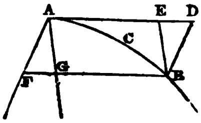
Therefore the points *b* and *d* ultimately coincide, and the lines *Ab*, *Ad*, are equal and coincident. (Art. 23.)

Therefore the curve *Acd* is ultimately equal to and coincident with either of the lines *Ab*, *Ad*. (Art. 24.)

Therefore, since it has been proved that the lines *ACB*, *AB*, and *AD* are always in the same ratio as *Acb*, *Ab*, and *Ad*, the line *ACB* is ultimately equal to and coincident with either of the lines *AB* and *AD*.

That is, when the arc *ACB* vanishes by the approach of *B* to *A*, the curve is ultimately in a ratio of equality to the chord *AB*, and the tangent *AD*.

Cor. 1. If through the second point *B*, be drawn *BF* parallel to the tangent, and *AF* a secant through *A* parallel to *BD*, *BF* will *ultimately* be equal to the arc *ACB*; for *BF* is equal to *AD*.



Cor. 2. If several secants be drawn through *B*, as *BD*, *BE*, or through *A*, as *AF*, *AG*, all the *abscissas* (or lines cut off) *AD*, *AE*, *BF*, *BG*, will *ultimately* be equal to the arc, and therefore to each other.

Cor. 3. Hence these lines, in the ultimate form of the hypothesis, (namely on the supposition that *B* approaches indefinitely near to *A*,) may be taken for each other.

N. B. The secants must all make with the tangent angles which are ultimately finite.

Prop. VIII. (Newton, Lemma viii.)

If two lines, drawn through the first and second points in the curve, make with the chord, the curve, and the tangent at the first point three triangles;—

And if the second point approach indefinitely near to the first;—

The three triangles are ultimately similar as to form, and in a ratio of equality.

Let *AR*, *BR* (Fig. p. 31) be the two lines; and making the same construction as in last Proposition, let *dbr* be parallel to *DBR*.

Let *Ad* remain finite when *B* approaches indefinitely to *A*; therefore *Ab*, *Acb*, *Ad* coincide; and the three triangles *rAb*, *rAd*, *rAcb*, are *ultimately* identical as to form, and in a ratio of equality. Therefore the three triangles *RAB*, *RAD*, *RACB*, (which are always similar and proportional to the others by Prop. 6.) are *ultimately* similar as to form, and in a ratio of equality.

Cor. Hence these triangles, in the ultimate form of the hypothesis, and in all reasonings founded upon it, may be taken for each other.

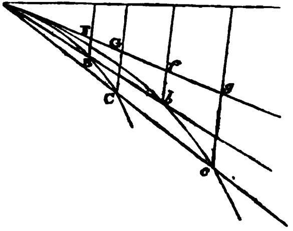
Prop. IX. (Newton, Lemma ix.)

If through a fixed point in a curve there be drawn a secant, and if through two variable points of the curve there be drawn *ordinates* (that is, parallel lines) meeting the secant, and making two triangles (each having a curvilinear side;)—

And if these two variable points approach at the same time indefinitely near the fixed point;—

The two triangles will *ultimately* be in the ratio of the squares of their corresponding sides.

Let *A* be the fixed point, *B*, *C*, the variable points, *BD*, *CE* the ordinates to the secant *AE*. Let *c* be taken in *AC* produced, and let the curve *Abc* be always similar to *ABC*, *ABb* being a straight line. Let also ordinates *bd*, *ce* be drawn meeting the secant; and let *AFGfg* be a tangent to *ABC*, and therefore to *Abc*. (Prop. 6.)

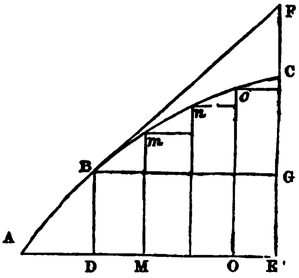


Let Ae remain constant while C and B approach indefinitely to A. The angle cAg or CAG will vanish; the triangles Abd, Ace will *ultimately* coincide with the triangles Afd, Age; and the areas Abd, Ace will ultimately be in the same ratio as the areas Afd, Age; namely, by similar triangles, as the squares of Ad, Ae. But the areas ABD, ACE are proportional to the areas Abd, Ace; and AB, AD are proportional to Ab, Ad, by Prop. 6. Hence it follows that ABD, ACE, are ultimately as the squares of AD, AE.

Prop. X.

If a body move with any finite (variable) velocity, and if there be taken a curve of which the abscissas represent the times, and the rectangular ordinates the velocities of the body at the times respectively; the areas of the curvilinear space intercepted by the ordinates will be proportional to the space described in the respective times.

Let the ordinates of the curve BC, from D to E, represent the velocities, at the times which are represented by the abscissas from AD to AE; the area $BDEC$ will be proportional to the space described in the time DE.



Let the time DE be divided into any small portions, DM, MN, NO, OE, and let the ordinates Mm, Nn, Oo, be drawn, and the parallelograms BM, mN, nO, oE, be completed. If the body were to move during the portion of time DM with the velocity which it has at the beginning of that portion, the space would be proportional to the velocity and time jointly; and therefore proportional to the product DM and DB, or to the rectangle BM. In like manner, if the body were to move during the portion of time MN with the velocity which it has at the beginning of that portion, the space described in that portion would be proportional to the rectangle mN; and similarly for the other portions.

Therefore if the body were to move during each of the portions of time with the velocity which it has at the beginning of that portion, the whole space described would be proportional to the sum of the rectangles BM, mN, nO, oE.

But if the number of these portions be increased indefinitely, and the magnitude of each portion consequently increased indefinitely, the space thus described will have for its limit, the space described by the variable velocity, in the whole time, by Article 14.

And on the same supposition, the sum of the rectangles will have for its limit the curvilinear area $BDEC$, by Prop. 3.

Therefore the space described in the whole time, will be proportional to the curvilinear area $BDEC$. Q. E. D.

Cor. 1. The velocity at any point is the limit of the quotient of the increment of the space, divided by the corresponding increment of the time, when these increments vanish.

For, completing the parallelogram $BDEG$,

$$BD \times DE = BDEG;$$

$$\text{and } BD = \frac{BDEG}{DE} = \frac{BDEC}{DE} \times \frac{BDEG}{BDEC};$$

and when DE vanishes, the limit of $\frac{BDEG}{BDEC}$ is 1, by Cor. 4, to Prop. 3. Therefore BD is equal to the limit of $\frac{BDEC}{DE}$ when DE vanishes.

But BD represents the velocity, DE the increment of the time, and $BDEC$ the increment of the spaces.

$$\text{Therefore velocity} = \text{limit of } \frac{\text{increment of space}}{\text{increment of time}}.$$

PROP. XI.

If a body move by the action of any finite (variable) force, and if there be taken a curve of which the abscissas represent the times, and the (rectangular) ordinates the force at the times respectively, the areas of the curvilinear spaces intercepted by the ordinates will be proportional to the velocities generated in the respective times.

COR. 1. The force at any point is the limit of the quotient of the increment of the velocity, divided by the corresponding increment of the time, when these increments ultimately vanish.

This Proposition and Corollary are proved from Article 15, in exactly the same manner as Prop. 10. from Article 14, substituting in the proof, "force" for "velocity", "velocity" for "space," and "velocity generated" for "space described."

COR. 2. The construction to Prop. 10. being made, and BF being drawn a tangent at B, meeting EC in F, the force in this construction at the time AD is represented by the fraction $\frac{FG}{BG}$.

For by Cor. 1, the force at the time AD is equal to the limit of $\frac{\text{increment of velocity}}{\text{increment of time}}$; that is, it is equal to the limit of $\frac{CG}{DE}$, when DE vanishes. But when DE vanishes, C comes up to B, and by Prop. 8, the limit of $\frac{CG}{DE}$ is $\frac{FG}{DE}$. Therefore, $\frac{FG}{DE}$ or $\frac{FG}{BG}$ represents the force.

COR. 3. In this construction, if the force be finite, $\frac{FG}{BG}$ is finite, and the curve makes a finite angle with the abscissa.

PROP. XII. (NEWTON, Lemma x.)

If a body be urged by any finite force, the spaces described from the beginning of the motion, when they are taken indefinitely small, are ultimately as the squares of the times.

Let the construction of Prop. 10. be made. And since the spaces are taken from the beginning of the motion, when the velocity is nothing, the curve, the ordinates of which represent the velocities, will meet the abscissa; and since the force is finite, will meet the abscissa at a finite angle, by Cor. 2. to Prop. 11.

Let ABC (see fig. p. 33) be this curve, AD, AE being any two times; therefore by Prop. 10, the areas ADB, AEC will represent the spaces described from the beginning of the motion. But by Prop. 9, these areas are *ultimately* (when AD, AE are indefinitely diminished,) as the squares of the AD and AE. Therefore, the spaces described from the beginning of the motion, when the times are indefinitely diminished, are as the squares of the times. Q. E. D.

COR. 1. If the force be constant, the velocity DB increases proportionally to the time AD, and ABC is a straight line.

COR. 2. Hence the space described in any time by the action of a constant force from rest, is equal to half the space described in the same time with the velocity last acquired.

For the space described in the time AE is represented by the triangle AEC, while the space described in the time AE with the last acquired velocity EC, is the rectangle $AE \times EC$, which is double the triangle.

COR. 3. When the force is not constant, the space described from the beginning of the motion is *ultimately* half the space described in the same time with the last acquired velocity.

For both these spaces are ultimately equal to what they would have been had the force been constant, as appears by the figure.

Cor. 4. If f represent a constant force, measured by the velocity which it would produce in a time 1, ft will be the velocity produced in time t, and therefore, by Cor. 2, $\frac{1}{2}ft^2$ will be the space described from each in a time t.

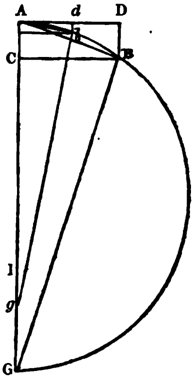
Cor. 5. Since s the space $= \frac{1}{2}ft^2$, and $v = ft$, $v^2 = 2f \cdot \frac{1}{2}ft^2$, whence $v^2 = 2fs$.

Prop. XIII. (Newton, Lemma xi.)

In curves of finite curvature, the subtense of the angle of contact is ultimately as the square of the conterminous arc.

Case 1. When the subtense is perpendicular to the tangent.

Let AB be the arc, Ab any other arc; let BD, bd, both perpendicular to AD, be subtenses; and let BG, bg be drawn perpendicular to the chords AB, Ab, meeting in G, g, the line AGg perpendicular to AD. The semi-circle described on AG will always pass through the point B.



The curvature of the curve at A is the same as the curvature of the circle which is the limit of the circle ABG (Art. 11). But the curvature of the curve at A is finite; therefore the curvature of the limit of the circle ABG is finite; therefore (Art. 12.) the diameter of the limit of the circle ABG is finite; therefore AG has a finite limit.

Let this limit be AI. Therefore AI is the ultimate value of AG and of Ag.

The triangles BDA, ABG are similar; (for they have right angles BDA, ABG, and equal angles DBA, BAG.)

$$\text{Therefore } GA : AB :: AB : BD;$$

$$\text{and therefore } AG \times BD = AB^2.$$

In like manner $Ag \times bd = Ab^2$.

$$\text{Hence, } \frac{AB^2}{Ab^2} = \frac{GA \times BD}{gA \times bd} = \frac{AG}{Ag} \times \frac{BD}{bd}.$$

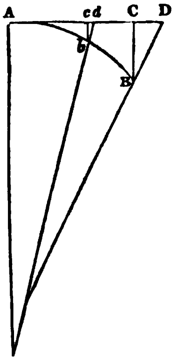
And ultimately AG, Ag are each equal to AI, and therefore equal to each other. (Art. 26.) Therefore ultimately $\frac{AG}{Ag} = 1$; and therefore ultimately $\frac{AB^2}{Ab^2} = \frac{BD}{bd}$.

$$\text{Therefore ultimately } \frac{BD}{bd} = \frac{(\text{chord } AB)^2}{(\text{chord } Ab)^2};$$

$$\text{but ultimately } \frac{\text{chord } AB}{\text{chord } Ab} = \frac{\text{arc } AB}{\text{arc } Ab}; \text{ by Prop. 7;}$$

$$\text{therefore ultimately } \frac{BD}{bd} = \frac{(\text{arc } AB)^2}{(\text{arc } Ab)^2};$$

therefore ultimately $BD : bd :: (\text{arc } AB)^2 : (\text{arc } Ab)^2$.



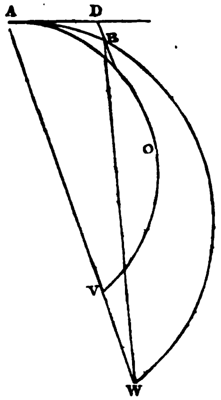
CASE 2. Let the subtenses BD, bd, be inclined to the tangent BD at equal angles. Then BC, bc, being perpendiculars as before, $BD : bd :: BC : bc$ by similar triangles; and $BC : bc :: (AB)^2 : (Ab)^2$ by the former case; therefore, in this case also, $BD : bd :: (AB)^2 : (Ab)^2$.

CASE 3. Let the subtenses BD, bd, be inclined to AD at an angle, variable according to any law, but ultimately finite.

Let BC, bc be drawn parallel to the ultimate position of the subtense BD. Then BD is to BC *ultimately* in a ratio of equality. In the same manner bd is to bc *ultimately* in a ratio of equality. Therefore *ultimately*

$BD : bd :: BC : bc :: AB^2 : Ab^2$, by the last Case.

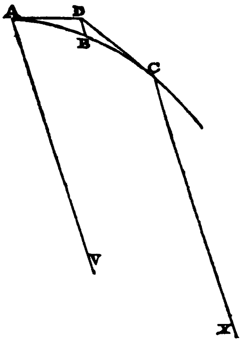
Cor. 1. If AV be a chord of the circle of curvature parallel to the subtense BD, ultimately $AV.BD = AD^2$. For in the circle ABW, the angle BWA is always equal to BAD, and BAW to ABD. Therefore the triangles ABD, WAB are similar; therefore $AW : AB :: AB : BD$, and $AW.BD = AB^2$.



And ultimately AW becomes AV, AB becomes equal to AD, and the circle becomes the circle of curvature AOV:

Therefore $AV.BD = AD^2$ ultimately.

Cor. 2. If DA, DC are two tangents meeting in D, *ultimately*, when C comes up to A, $DC = DA$.



Draw a subtense BD, and AV, CX the chords of curvature parallel to it. Then by the last Corollary,

$$AV.BD = DA^2,\ CX.BD = DC^2.$$

Therefore,

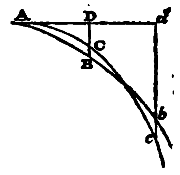
$$\begin{aligned} DC^2 : DA^2 &:: CX.BD : AV.BD \\ &:: CX \quad : AV \end{aligned}$$

and therefore ultimately $DC^2 : DA^2$ are in a ratio of equality, and $DC = DA$.

SCHOLIUM.

ON POINTS WHERE THE CURVATURE IS NOT FINITE.

1. Def. If AB, AC be two curves having a common tangent AD; and if DCB be the subtense; the ratio of the angles of contact BAD, CAD of the curves AB, AC, is measured by the *ultimate* ratio of the subtenses BD, CD.



2. **Prop.** If the subtense DB be as AD^2, and DC as AD^3, the angle of contact of AC is infinitely less than that of AB.

For let dbc, parallel to DCB, be drawn; then

$$\begin{array}{rlllll}
 & DB & : db & :: AD^2 & : Ad^2 & \text{by hyp.} \\
\text{Whence} & DB^3 & : db^3 & :: AD^6 & : Ad^6 & \\
\text{also} & DC & : dc & :: AD^3 & : Ad^3 & \text{by hyp.} \\
\text{whence} & DC^2 & : dc^2 & :: AD^6 & : Ad^6 & \\
\text{therefore} & DB^3 & : db^3 & :: DC^2 & : dc^2 & \\
\text{and} & DB^3 & : db^3 & :: DC^3 & : DC.dc^2 & \\
\text{or} & DB^3 & : DC^3 & :: db^3 & : DC.dc^2. &
\end{array}$$

Let Ad, and therefore db, dc, remain finite, and let AD be indefinitely diminished. Then *ultimately* DC vanishes, and the ratio $db^3 : DC.dc^2$ becomes infinitely great. Therefore the ultimate ratio of $DB^3 : DC^3$ is infinitely great; and therefore so also is the ratio of $DB : DC$, and that of the angle of contact BAD to CAD.

3. **Prop.** If the subtense DB (fig. above) be as AD^m and DC as AD^n, where m is less than n, the angle of contact of AC with AD is infinitely less than that of AB.

For because

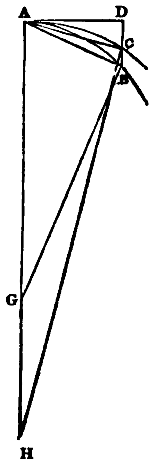
$$\begin{array}{rllll}
 & DB & : db & :: AD^m & : Ad^m,\ DB^n : db^n :: AD^{mn} : Ad^{mn} \\
\text{so} & DC & : dc & :: AD^n & : Ad^n,\ DC^m : dc^m :: AD^{mn} : Ad^{mn} \\
\text{hence} & DB^n & : db^n & :: DC^m & : dc^m. \text{ Multiply by } DC^{n-m} \\
 & DB^n & : db^n & :: DC^n & : DC^{n-m}dc^m \\
\text{or} & DB^n & : DC^n & :: db^n & : DC^{n-m}dc^m.
\end{array}$$

When D approaches indefinitely to A, DC vanishes, and therefore DC^{n-m} vanishes, because $n-m$ is positive; but db, dc remain finite. Therefore the ratio $DB^n : DC^n$ becomes infinite; and therefore the ratio $DB : DC$, and that of the angles of contact BAD to CAD.

4. **Cor.** Hence if DB be made successively to vary as AD^2, AD^3, AD^4, AD^5, AD^6, AD^7, &c. we shall have an infinite series of angles of contact each infinitely less than the pre-

ceding; and if we take DB successively as AD^2, $AD^{\frac{3}{2}}$, $AD^{\frac{4}{3}}$, $AD^{\frac{5}{4}}$, $AD^{\frac{6}{5}}$, $AD^{\frac{7}{6}}$, &c. we shall have another infinite series of angles of contact, of which the first is of the same kind with the angles of contact in circles (Prop. 13), the next infinitely greater, and each angle infinitely greater than the preceding. Moreover between any two of these angles we may insert a series of mean terms, infinite in each direction, of which each is infinitely less or infinitely greater than the preceding one. Thus between the terms AD^2 and AD^3 we may insert the series $AD^{\frac{13}{6}}$, $AD^{\frac{11}{5}}$, $AD^{\frac{9}{4}}$, $AD^{\frac{7}{3}}$, $AD^{\frac{5}{2}}$, $AD^{\frac{8}{3}}$, $AD^{\frac{11}{4}}$, $AD^{\frac{17}{6}}$, &c. And again, between any two of the angles of this series, we may insert a new series of intermediate angles, differing from each other by infinite intervals. And there is no limit to this progression of properties.

5. **Def.** When the *circle of curvature* is infinitely *large*, the *curvature* is considered as infinitely *small*, and conversely.



6. **Prop.** If the angle of contact of AC be infinitely less than that of AB, the diameter of the circle of curvature of the former curve will be infinitely greater than that of the latter.

Let BCD be perpendicular to AD, BG to AB, CH to AC.

$$\text{Then } AG \cdot BD = AB^2,\ AH \cdot CD = AC^2.$$

And *ultimately* $AB^2 = AC^2$, therefore ultimately

$$AG \cdot BD = AH \cdot CD;$$

$$\text{or ultimately } AG : AH :: CD : BD.$$

If therefore CD be infinitely less than BD ultimately, AH will ultimately be infinitely greater than AG: that is, the diameter of curvature of AC will be infinitely greater.

7. **Cor. 1.** The curvature of a curve is infinitely less than that of another curve, when the circle of curvature is infinitely

greater. Hence propositions concerning the curvatures agreeing with those already proved concerning the angles of contact (Arts. 2, 3, 4 of this Scholium) are true.

8. COR. 2. The curvature of a circle and of any conic section is finite. Hence Prop. 13 is true of all points of such curves; but not of points of curves where the curvature is infinitely greater or infinitely less than that of a circle.

SCHOLIUM.

ON CERTAIN AXIOMS AND PROPOSITIONS IN MECHANICS.

1. IN the elementary parts of the science of Mechanics, certain axioms are employed of the following kind: (see Mechanical Euclid, Book I. Def. 14, 15, 16): that a heavy line may be supposed to be constituted of a row of heavy points; that a heavy plane may be supposed to be constituted of lines; and that a heavy solid may be supposed to be constituted of a series of planes. But these suppositions are not consonant to geometrical notions, since points cannot make up a line, or lines a plane, or planes a solid. To make the reasonings satisfactory, in which such axioms are employed, we must introduce the notion of Limits; and must employ the following axioms, by means of which the reasonings, with a slight alteration, will be valid.

2. AXIOM. A heavy body being supposed to be divided into particles, and each particle collected at a point in the particle itself, the mechanical effect of all the particles is, *ultimately*, (when the magnitude of each particle is diminished indefinitely,) equal to the effect of the body.

3. In nearly the same manner the reasoning may be made satisfactory concerning fluids. In order to avoid the use of limits in the elementary propositions of Hydrostatics, we employ an axiom of this kind: (Mechanical Euclid, B. II. Axiom 7.): that finite pressures and weights which are equal, except so far as the effect of a single particle is concerned, are absolutely equal. The propositions which are proved by means of this axiom, may be more rigorously enunciated and

proved by taking the limiting propositions; namely, when the particle becomes indefinitely small. For example, we shall have, instead of Prop. 3. (Mech. Euclid, B. II.) this Proposition;—In a fluid at rest, the pressures upon a particle in different directions are *ultimately* (when the particle is indefinitely diminished) equal. And the whole of the reasoning may be accommodated to this statement of the Proposition.

4. In the doctrine of motion also (Mech. Euclid, B. III.) certain Propositions are assumed as axioms, in order to avoid the introduction of Limits. For instance, Axiom 3:—If two bodies move, having their velocities at every instant in a constant ratio, the spaces described in any time by one body and by the other will be in the same ratio.

This Proposition may be demonstrated by means of Article 14 of this Book. For if the time be divided into any small portions, and if the two bodies move, retaining for each of these portions, the velocities which they have at the beginning of each portion; the spaces in the corresponding portions of each motion will be in the constant ratio of the velocities of the two bodies; and hence the whole space described by one body and by the other on this supposition will be in the same ratio. But the spaces described by the bodies on this supposition, have for their limits (when the portions are indefinitely diminished) the spaces described with the constantly varying velocities. (Art. 14 of this Book.) Hence the latter spaces are in the constant ratio, as was asserted.

5. In exactly the same manner may be proved the Proposition assumed in Axiom 9, (Mech. Euclid, B. III. Prop. 13.) —that if two bodies move so that their accelerating forces at every instant are in a constant ratio, and are in the direction of the motion, the velocities added or subtracted in any time are in the ratio of the accelerating forces. This would follow from Art. 15 of this Book.

6. The other Proposition stated as Axiom 8, (Mech. Euclid, B. III.) would be proved nearly in the same manner.

BOOK III.

CONIC SECTIONS.

1. The Conic Sections which we here consider are the Parabola, Ellipse, and Hyperbola.

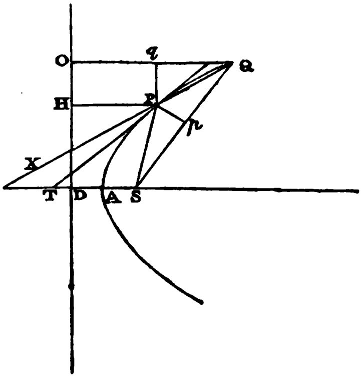
THE PARABOLA.

The *Parabola* is a line in which every point is equally distant from a given point (the *focus*), and a given straight line (the *directrix*).

It has appeared (B. I. Art. 13), that in this curve if A be the *vertex*, AM the *abscissa*, PM the *ordinate*, S the *focus*, $4AS.AM = PM^2$. Also $EA = SA$.

Prop. I.

In the parabola the tangent at any point bisects the angle made by two lines, one drawn to the focus, and one, perpendicular to the directrix.



Let P be any point in the parabola, and Q a point near it. Take in SQ, $Sp = SP$, and draw QO perpendicular and Pq parallel to the directrix.

By the property of the parabola, $SQ = QO$ and $SP = PH$; hence

$$SQ - SP = QO - PN,$$

$$\text{or } Qp = Qq.$$

Also, since $Sp = SP$, the angles SPp, SpP are ultimately equal, and when PSQ vanishes, are each ultimately a right angle. Therefore PpQ is ultimately a right angle; also PqQ is a right angle.

Hence the ultimately right-angled triangles PpQ, PqQ, which have a common side PQ, and sides Qp, Qq equal, are ultimately equal in all respects*. Therefore ultimately the angles $PQp = PQq$.

But ultimately

$$PQp \text{ or } PQS = SPX, \text{ and } PQq \text{ or } PQO = XPH;$$

wherefore ultimately $XPS = XPH$.

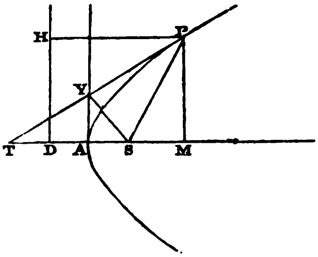
But the ultimate position of QPX, when Q comes to P, is a tangent at P. Therefore if TP be a tangent,

$$SPT = HPT.$$

Prop. II.

In the parabola $TM = 2AM$.

For since $SPT = HPT = PTS$, $ST = SP = PH = DM$, DH being the directrix.



Hence taking away SD, $DT = SM$, and adding $DA = SA$, $TA = AM$, whence $TM = 2AM$.

Cor. 1. If SY be perpendicular on PT, $PY = YT$, because $SP = ST$. Hence AY is perpendicular to AM.

Cor. 2. The angles ASY, YSP are equal; hence

$$AS : SY :: AS : SP \text{ and } SY^2 = AS \cdot SP.$$

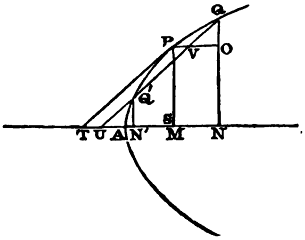
---

* For ultimately $Pq^2 = PQ^2 - Qq^2 = PQ^2 - Qp^2 = Pp^2$, and $Pq = Pp$.

PROP. III.

If QV be parallel to the tangent at P, and PV parallel to AM, $4SP.PV = QV^2$.

Let QV meet the axis in U, and let the ordinates QN, PM be drawn, and let PV meet QN in O.



By the property of the parabola, $4AS.AM = PM^2$, whence $\frac{AM}{PM} = \frac{PM}{4AS}$, and $\frac{2AM}{PM} = \frac{2PM}{4AS}$.

But by similar triangles

$$\frac{NU}{MT} = \frac{QN}{PM} = \frac{2QN}{2PM}.$$

Hence, compounding, (since $MT = 2AM$) $\frac{NU}{PM} = \frac{2QN}{4AS}$,

$$\text{and } 4AS.NU = 2QN.PM.$$

Also $4AS.AN = QN^2$ and $4AS.AT = PM^2$; because $AT = AM$.

Hence $4AS.NT = QN^2 + PM^2$: and we have already proved that $4AS.NU = 2QN.PM$; hence, subtracting

$$4AS.TU = QN^2 - 2QN.PM + PM^2 = (QN - PM)^2$$

or $4AS.PV = QO^2$.

Again, $\frac{QV^2}{QO^2} = \frac{TP^2}{PM^2} = \frac{ST^2}{SY^2}$ (Cor. 2. Prop. 2, and its fig.)

$$= \frac{SP^2}{AS.SP} = \frac{SP}{AS} = \frac{4SP.PV}{4AS.PV}:$$

And $QO^2 = 4AS.PV$: therefore $QV^2 = 4SP.PV$.

COR. 1. If QV meet the parabola again in Q, $Q'V = QV$.

For $Q'V^2 = 4AS.PV$, by the same reasoning as before

$$= QV^2, \text{ where } Q'V = QV.$$

Cor. 2. The line drawn through P parallel to the axis bisects all the lines drawn parallel to the tangent at P.

The line PV is called a *diameter* of the parabola, to the ordinate QQ' and others parallel to QQ'; and $4SP$ is the *parameter* to this diameter, or to the point P.

THE ELLIPSE.

The *Ellipse* is a line in which every point has the sum of its distances from two given points (the *foci*) equal to a constant quantity.

It has appeared (B. I. Arts. 14, 15, 16, 17) that the sum of the distances from the foci is equal to the *axis major*, or line passing through the foci and limited by the ellipse.

Also that e, the *ratio* of the distance between the foci to the major axis $2a$, is the *excentricity*; and if $b^2 = a^2(1 - e^2)$ b is the *axis minor*.

Also, that if a semi-circle be described on the axis major, the ordinates of the semi-circle and of the ellipse are in a constant ratio.

The point or line in the circle corresponding to any point in the ellipse has been termed the *circular projection* of the point or line.

The middle point of the major axis is the *centre of* the ellipse.

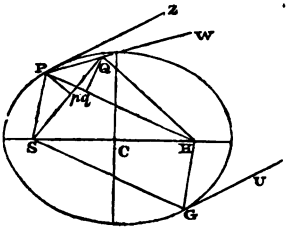
A *diameter* is any line drawn through the centre of the ellipse; its *vertex* is the point where it meets the curve; and *ordinates* to this diameter are lines in the curve parallel to the tangent at the vertex.

A diameter *conjugate* to another is a diameter parallel to a tangent at the vertex of the other.

Prop. III.

In the ellipse, the tangent at any point makes equal angles with two lines drawn to the foci.

Let P be any point in the ellipse; Q a point near to P; S, H the two foci. And let Sp be taken in SQ equal to SP, and Hq in HP equal to HQ.



By the property of the ellipse, $SP + PH = SQ + QH$; whence, $PH - QH = SQ - SP$, or $Pq = Qp$.

Also, since $Sp = SP$ the angles SPp, SpP are equal, and when PSQ vanishes are each ultimately a right angle. Therefore, PpQ is ultimately a right angle; and for the same reason PqQ is ultimately a right angle.

Hence, the ultimately right-angled triangles PpQ, PqQ, which have a common side PQ, and the sides Pq, Qp ultimately equal, are ultimately equal in all respects. Therefore ultimately the angles $PQp = QPq$.

But ultimately,

PQp or $PQS = XPS$*, and QPq or $QPH = WPH$; therefore ultimately $XPS = WPH$.

But the ultimate position of QPX, when Q comes to P, is a tangent at P. (B. II. Art. 7.) Therefore if ZPY be a tangent, $YPS = ZPH$.

Cor. 1. The tangents at the two extremities of any diameter are parallel.

For if $PHGS$ be a parallelogram, $SG + HG = SP + HP$, whence G is a point in the ellipse; and it is easily seen that the tangent GU is parallel to PZ.

Cor. 2. By the same figure, $CG = CP$, or the center bisects any diameter.

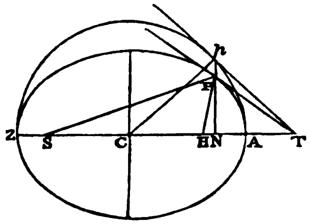
Prop. IV.

A tangent to an ellipse at any point, and the tangent at the circular projection of the point, meet the axis in the same point.

---

* In the diagram, the point Q should be in the curve, WPX should be the line WP produced, and ZPY should be the line ZP produced.

Since PT makes equal angles with PH and SP produced,



$$ST : HT :: SP : HP, \text{ (Euc. VI. A.)}$$

$$\text{Hence, } ST + HT : ST - HT :: SP + HP : SP - HP,$$

$$\text{or } 2CT : SH :: 2AC : SP - HP.$$

$$\text{But } SH : 2AC :: SP - HP : SM - HM^{*}.$$

Therefore compounding the last two proportions,

$$2CT : 2AC :: 2AC : SM - HM (= 2CN).$$

$$\text{And } CT : AC :: AC : CM.$$

Let ApZ be the circular projection of the ellipse APZ, p the circular projection of P; and join pT. Since $CM : CA :: CA : CT$, we have $CM : Cp :: Cp : CT$. Therefore the triangles CMp, CpT, having a common angle pCT, are similar. Therefore the angle CpT is equal to CNp, and is a right angle. Therefore CpT is a tangent to the circle at p.

PROP. V.

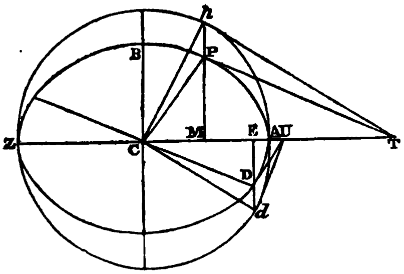
In an ellipse, a diameter and its conjugate diameter have their circular projections at right angles to each other.

Let CP be a semi-diameter; CD, parallel to the tangent PT, its conjugate. Let $ApZd$ be the circular projection of the ellipse $APZD$; p, d being the circular projections of

---

* For $SP^2 - SN^2 = PM^2 = HP^2 - HM^2$; whence $SP^2 - HP^2 = SM^2 - HM^2$: therefore $(SP + HP)(SP - HP) = (SN + HM)(SN - HM)$ or $2AC.(SP - HP) = SH(SM - HM)$, whence the proportion.

P, *D*; and let PT, pT, DU, dU be tangents to the ellipse and to the circle. (Prop. 4.)



Since Mp, Ed are perpendicular to CA, by similar triangles $CM \,.\, MT = Mp^2$, hence (B. I. Art. 17)

$$CM \,.\, MT : MP^2 :: Ed^2 : ED^2,$$

and $CM \,.\, MT : Ed^2 :: MP^2 : ED^2$;

whence $CM \,.\, MT : Ed^2 :: MT^2 : CE^2$, by parallels PT, CD,

whence $CM \,.\, MT : MT^2 :: Ed^2 : CE^2$,

or $CM^2 : CM \,.\, MT :: Ed^2 : CE^2$;

therefore $CM^2 : Mp^2 :: Ed^2 : CE^2$,

and $CM : Mp :: Ed : EC$.

Hence the triangles CMp, dEC are similar, the angle dCE is the complement of MCp, and pCd is a right angle. Q.E.D.

Cor. 1. If CD be conjugate to CP, CP is conjugate to CD.

Cor. 2. $CM = Ed$, and $CE = Mp$.

Cor. 3. $CP^2 + CD^2 = AC^2 + BC^2$.

$$\text{For } CP^2 + CD^2 = CM^2 + MP^2 + CN^2 + ED^2$$
$$= CM^2 + MP^2 + Mp^2 + ED^2$$
$$= Cp^2 + MP^2 + ED^2.$$

But $$\frac{MP^2 + ED^2}{BC^2} = \frac{MP^2}{BC^2} + \frac{ED^2}{BC^2} = \frac{Mp^2}{AC^2} + \frac{Ed^2}{AC^2}$$

$$= \frac{Mp^2 + CM^2}{AC^2} = 1.$$

Hence $MP^2 + ED^2 = BC^2$, and $CP^2 + CD^2 = AC^2 + BC^2$.

COR. 4. If $CM = x$, $Ed = x$, $ED^2 = \frac{b^2}{a^2}x^2$,

and $CE^2 = Mp^2 = a^2 - x^2$.

Hence $CD^2 = a^2 - x^2 + \frac{b^2}{a^2}x^2 = a^2 - \frac{a^2 - b^2}{a^2}x^2 = a^2 - e^2x^2$.

COR. 5. In like manner $CP^2 = b^2 + e^2x^2$.

COR. 6. $SP \cdot HP = CD^2$. For, as in Art. 14, Book I, $SP = a + ex$, $HP = a - ex$:

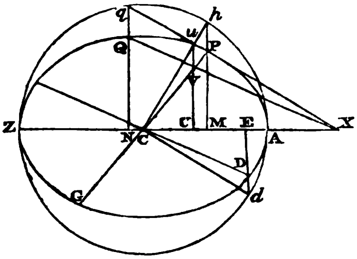
hence $SP \cdot HP = a^2 - e^2x^2 = CD^2$, by Cor. 4.

PROP. VI.

In an ellipse, if QV be an ordinate to a diameter CP, CD the semi-conjugate diameter to CP;

$$PV \cdot VG : QV^2 :: CP^2 : CD^2.$$

Let $APZD$ be an ellipse, GP a diameter, QV an ordinate to the diameter, CD the semi-diameter conjugate to CP. Let $ApZd$ be the circular projection of the ellipse; p, q, d the circular projections of points P, Q, D. Let UV, perpendicular to AZ, meet Cp in v; and join qv.



By Prop. 5, Cd is a perpendicular to Cp. Also

$Uv : UV :: Mp : MP$, which is $:: Nq : NQ$.

Hence qv, QV will meet NU in the same point X. And since VX is parallel to CD (because QV is an ordinate), and that $UV : Uv :: ED : Ed$, it is easily seen that Xv is parallel to Cd; and therefore qv is perpendicular to Cp: and hence $Cp^2 - Cv^2 = Cq^2 - Cv^2 = qv^2$.

Now $CP^2 : CV^2 :: Cp^2 : Cv^2$;

$$\therefore\ CP^2 - CV^2 : Cp^2 - Cv^2 :: CP^2 : Cp^2;$$

$$\therefore\ CP^2 - CV^2 : \quad qv^2 \quad :: CP^2 : Cp^2.$$

$$\text{Also}\quad qv^2 : QV^2 :: Cd^2\,(Cp^2) : CD^2;$$

∴ compounding the two last propositions,

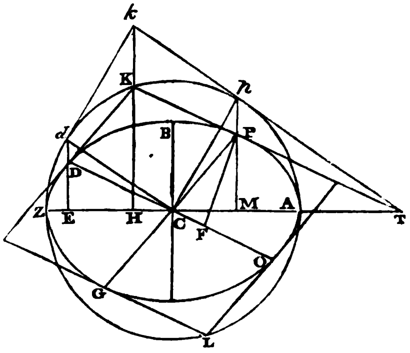
$$CP^2 - CV^2 : QV^2 :: CP^2 : CD^2,$$

$$\text{or}\quad PV.VG : QV^2 :: CP^2 : CD^2.$$

Prop. VII.

The parallelograms made by drawing tangents at the extremities of two conjugate diameters of an ellipse are all equal in area.

Let $APDZ$ be an ellipse, $ApdZ$ its circular projection,



CP, CD two semi-diameters conjugate to each other: KL a parallelogram made by drawing tangents at the extremities of the diameters PC, DC.

Draw HK perpendicular to AZ, meeting in k the tangent of the circular projection at p. Therefore since the tangent of the ellipse and of its circular projection meet AZ in the same point T, we have $HK : Hk :: MP : Mp$, that is, $HK : Hk :: BC : AC$. For the same reason the tangent at D will meet Hk in a point determined by the same proportion. Therefore the two tangents at p and d meet HK in the same point k. And Cp is at right angles to Cd and equal to it; therefore $Cpkd$ is a square.

Now the triangles THK, THk are as their bases $HK : Hk$; that is,

$$THK : THk :: BC : AC.$$

Also $TMP : TMp :: BC : AC$; hence the differences are in the same proportion; that is,

$$\text{trapezium } MPKH : MpkH :: BC : AC.$$

In like manner, trapezium $EDKH : EdkH :: BC : AC$.

Also triangle $CPM : CpM :: BC : AC$,

and triangle $CDE : CdE :: BC : AC$.

Add together the two former of these four sets of proportionals, and subtract the two latter, and we have

$$CPKD : Cpkd :: BC : AC.$$

$$\text{Whence, } CPKD : Cpkd :: AC.BC : AC^2.$$

But $Cpkd$ is equal to Cp^2 or AC^2. Therefore $CPKD$ is equal to $AC.BC$.

The parallelogram KL is four times the parallelogram $CPKD$. Therefore the parallelogram $KL = 4AC.BC$, and is constant.

COR. If PF be drawn perpendicular on DC, the parallelogram $CPKD$ is equal to $CD.PF$. Therefore

$$CD.PF = AC.BC.$$

THE HYPERBOLA.

The Hyperbola is a line in which every point has the difference of its distances from two given points (the *foci*), equal to a constant quantity.

It has appeared (B. I. Arts. 19, 20, 23) that the difference of the distances from the foci is equal to the *axis major*, or line passing through the foci, and limited by the hyperbola:

Also, that if e be the ratio of the distance between the foci to the axis major $2a$, e is the *excentricity;* and if $b^2 = (e^2 - 1)a^2$, b may be called the *axis minor* (although b may be greater than a).

Also, that if CM, MP be x and y, $y^2 = \frac{b^2}{a^2}(x^2 - a^2)$.

Also, that there is an *opposite hyperbola* derived from the same property.

Also, that these two curves have common *asymptotes;* and that the asymptotes make with the axis an angle θ, such that $\tan\theta = \frac{b}{a}$.

If we construct other hyperbolas in which the major and minor semi-axes are b and a respectively, these will be the *conjugate hyperbolas.*

The conjugate hyperbolas have the same asymptotes as the original hyperbolas.

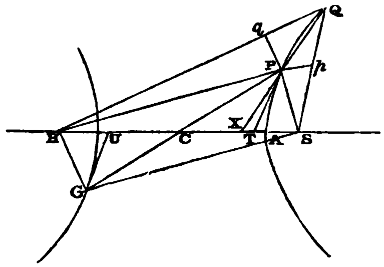
Diameter, *vertex*, *ordinates*, are defined as in the ellipse.

A *diameter conjugate* to another is a diameter of the conjugate hyperbola, parallel to the tangent at the vertex of the other.

PROP. VIII.

A tangent at any point of the hyperbola bisects the angle made by two lines drawn to the foci.

Let P be any point in the hyperbola; Q a point near to P; S, H, the two foci. And let Sp be taken in SQ, equal to SP, and Hq in HQ equal to HP.



By the property of the hyperbola,

$$HP - SP = HQ - SQ;$$

whence

$$SQ - SP = HQ - HP, \text{ or } Qp = Qq.$$

Also since $Sp = SP$, the angles SPp, SpP are equal, and when PSQ vanishes, are each ultimately a right angle. Therefore PpQ is ultimately a right angle; and also PqQ for the same reason. Hence the ultimately right-angled triangles PpQ, PqQ, which have a common side PQ, and sides Qp, Qq ultimately equal, are ultimately equal in all respects. Therefore ultimately the angles $PQp = PQq$. But ultimately PQp or $PQS = XPS$, and PQq or $PQH = XPH$: wherefore ultimately $XPS = XPH$. But the ultimate position of QPX, when Q comes to P, is a tangent at P. Therefore if TP be a tangent, $SPT = HPT$.

Cor. 1. The tangents at two extremities of any diameter are parallel. For if PG be a parallelogram, $GS - GH = HP - SP$, whence G is a point in the hyperbola; and it is easily seen that the tangent GU is parallel to the tangent PT.

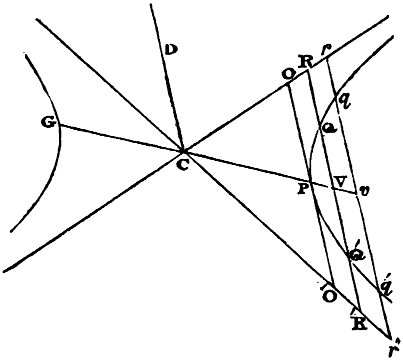
Cor. 2. By the same figure $CG = CP$, or the centre bisects any diameter.

Prop. IX.

Any diameter of a hyperbola bisects all its ordinates.

Let $RQQ'R'$ be a line cutting the hyperbola and its asymptotes; and V the bisection of RR'. Then (B. I. Art. 26) $QR \cdot QR' = Q'R \cdot Q'R'$; that is, $VR^2 - VQ^2 = VR'^2 - VQ'^2$; whence $VQ = VQ'$, and $QR = Q'R'$.

Let RR' move parallel to itself, till the points Q and Q' come together at P, and the line comes into the position OPO': then OPO' is a tangent; and, as before, we have $PO = PO'$.



Hence, joining PV, CPV is a straight line, and is a diameter. And QQ', being parallel to the tangent OO', is an ordinate; and QQ' is bisected by the diameter CV.

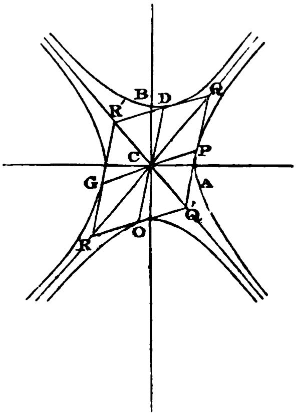
Q. E. D.

Cor. If $OP = PO'$, as in B. I. Art. 27, the diameter conjugate to CP is parallel to PO. And if ϕ, ψ be the angles which OO' and PC make with the axis major, $\tan\psi \cdot \tan\phi = \frac{b^2}{a^2}$, by that Article.

Prop. X.

In the hyperbola, if *CD* be conjugate to *CP*, *CP* is conjugate to *CD*.

Let the angle PCA be ψ, and let ϕ be the angle which the tangent at P makes with the axis. Also let $CP = r$, $PQ = s$. And let CB be the conjugate semi-axis to CA; $CA = a$, $CB = b$.



By Cor. to Prop. 9, the angles ϕ, ψ are connected by the equation

$$\tan\phi \tan\psi = \frac{b^2}{a^2}.$$

Now, if the line CD be drawn parallel to PQ, meeting the conjugate hyperbola in D, the angle DCB will be the complement of ϕ. And if the tangent at D make with AC an angle ψ', it will make with BC an angle which is the complement of ψ'. And applying the formula, with these angles, to the axis CB (that is, putting b for a and a for b), we have

$$\cot\psi' \cot\phi = \frac{a^2}{b^2}, \quad \text{or} \quad \tan\psi' \tan\phi = \frac{b^2}{a^2}:$$

whence $\psi' = \psi$, and the tangent at D is parallel to CP.

Cor. The tangents at P and D meet the asymptote in the same point Q.

For by Book I. Art. 29,

$$CP = r = \frac{ab}{\sqrt{(b^2\cos^2\psi - a^2\sin^2\psi)}};$$

$$PQ = s = \frac{ab}{\sqrt{(a^2\sin^2\phi - b^2\cos^2\phi)}}.$$

Now, since CD is parallel to PQ, if it meet the conjugate hyperbola in D, the angle DCB will be the complement of ϕ; and hence putting b for a and a for b, and complement of ϕ for ψ in the formula for CP, we have

$$CD = \frac{ab}{\sqrt{(a^2\sin^2\phi - b^2\cos^2\phi)}} = PQ;$$

Hence DQ is parallel to CP and therefore tangent at D, by the Proposition.

Prop. XI.

In the hyperbola, the parallelograms formed by tangents drawn at the vertices of any pair of conjugate diameters have all the same area.

If PG*, DO be conjugate semi-diameters the tangents at P and G are parallel, as also at D and K, by Prop. 8. Cor. 1. And the tangent at P is parallel to CD, and the tangent at D parallel to CP by Prop. 10.

---

* See Diagram to Prop. 10.

Let the tangent at P meet the asymptotes in Q, Q'; and let the tangent at D meet the asymptotes in Q, R'. (Prop. 10. Cor.) Then, by Art. 28, Book I. $CQ' = \frac{a^2 + b^2}{CQ}$.

But CR' will be found by putting a for b, and b for a. Therefore, $CR' = \frac{b^2 + a^2}{CQ}$; and $CR' = CQ'$. In the same manner if RQ', RR' be the tangents at K and G, $CR = CQ$.

Therefore the parallelogram $QQ'RR'$ is four times the triangle QCA';

or parallelogram $= 2\,QC \cdot CQ' \sin QCQ' = 2(a^2 + b^2)\sin 2\theta$.

But $\tan\theta = \frac{b}{a}$; hence $\cos\theta = \frac{a}{\sqrt{a^2 + b^2}}$; $\sin 2\theta = \frac{2ab}{a^2 + b^2}$; whence parallelogram $= 4ab$; which is constant.

Cor. 1. If PF be drawn perpendicular on CD, $PF \cdot CD = AC \cdot CB$.

Cor. 2. PD is parallel to $Q'R'$, and therefore to AB; and is bisected by CQ.

Prop. XII.

In the hyperbola $CD^2 - CP^2 = CB^2 - CA^2$.

The equation to the hyperbola and to the conjugate hyperbola are respectively,

$$y^2 = \frac{b^2}{a^2}(x^2 - a^2), \quad y^2 = \frac{b^2}{a^2}(x^2 + a^2).$$

For the former is the equation to the hyperbola by Book I. Art. 20. And since in the conjugate hyperbola we must put b for a, and a for b, x for y, and y for x, the equation is

$$x^2 = \frac{a^2}{b^2}(y^2 - b^2), \text{ whence } y^2 = \frac{b^2}{a^2}(x^2 + a^2).$$

Now, if x, x' be the abscissas of P, D, we have,

$$\tan\psi = \frac{y}{x} = \frac{b}{a}\frac{\sqrt{x^2 - a^2}}{x}; \text{ similarly, } \tan\phi = \frac{b}{a}\frac{\sqrt{x'^2 + a^2}}{x'}.$$

And as in Prop. 9, Cor. $\tan\phi \,.\, \tan\psi = \frac{b^2}{a^2}$;

therefore $\frac{\sqrt{x^2 - a^2}}{x} \,.\, \frac{\sqrt{x'^2 + a^2}}{x'} = 1$, and $x'^2 = x^2 - a^2$.

Now, $CP^2 = x^2 + \frac{b^2}{a^2}(x^2 - a^2) = e^2x^2 - b^2$, since $e^2 = \frac{a^2 + b^2}{a^2}$.

And $CD^2 = x'^2 + \frac{b^2}{a^2}(x'^2 + a^2) = e^2x^2 - a^2$.

Hence $CD^2 - CP^2 = b^2 - a^2$.

Cor. 1. $SP \,.\, HP = CD^2$.

For (B. I. Art. 18) $SP = ex - a$, $HP = ex + a$.

Prop. XIII.

In the hyperbola,*

$$PV \,.\, VG : QV^2 :: CP^2 : CD^2.$$

Let qq' be parallel to QQ'; then since by Book I. Art. 22.

$$PV \,.\, VG : QV \,.\, VQ' :: Pv \,.\, vG :: qv \,.\, vq'.$$

But by Prop. 9, $VQ' = QV$, and $vq' = qv$.

Therefore,

$$\begin{aligned} PV \,.\, VG : QV^2 &:: Pv \,.\, vG : qv^2, \\ &:: Cv^2 - CP^2 : (vr - rq)^2, \\ &:: Cv^2\left(1 - \frac{CP^2}{Cv^2}\right) : vr^2\left(1 - \frac{rq}{vr}\right)^2. \end{aligned}$$

---

* See Diagram to Prop. 9.

Now when Cv becomes very large compared with CP, rq becomes very small compared with vr; and ultimately, when Cv becomes indefinitely great, the fractions $\frac{CP^2}{Cv^2}$ and $\frac{rq}{vr}$ vanish. Therefore, ultimately,

$$PV.VG : QV^2 :: Cv^2 : vr^2.$$

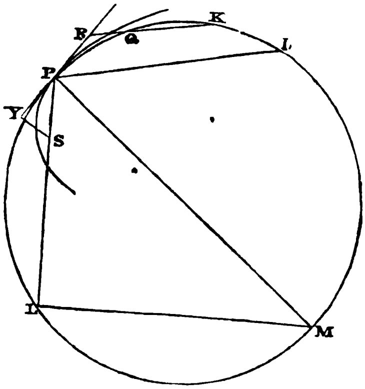
But $Cv^2 : vr^2 :: CP^2 : PO^2$ both ultimately and always

Therefore, $PV.VG : QV^2 :: CP^2 : PO^2$

$$:: CP^2 : CD^2, \text{ by Prop. 10. Cor.}$$

Prop. XIV.

In the parabola, to find the radius of curvature, the chord of curvature passing through the focus, and the chord of curvature parallel to the axis.



Let PQI be a circle which has a common tangent with the curve at P, and cuts the curve in Q*: then, by Book II. Art. 11, when Q comes to P, the ultimate value of this circle is the circle of curvature.

Let PI, QK be chords parallel to the axis of the parabola; QV parallel to PR, the tangent; S the focus. Then, by Prop. 3, $4SP.PV = QV^2$, or, since

$$PV = QR, \text{ and } QV = PR,$$

$4SP.QR = PR^2 = KR.QR$ by the property of the circle. Therefore $KR = 4SP$.

And when Q comes to P, KR becomes PI, the chord of curvature parallel to the axis. (B. I. Art. 11.) Therefore this chord $= 4SP$.

---

* The curve and circle should intersect at Q in the diagram.

Also, by Prop. 1, if PL be the chord drawn through the focus, PL and PI make equal angles with the tangent to the parabola at P; that is, with the tangent to the circle: hence they may easily be shewn to be equal, and PL the chord of curvature through the focus $= 4SP$.

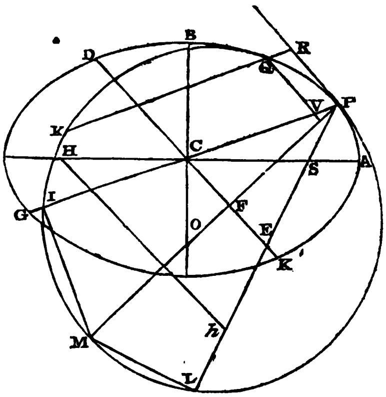
Again, if PM be the diameter of curvature, draw SY perpendicular on the tangent, and join LM; and it is easily seen that the triangles SPY, PML are similar.

Hence $PM = \frac{PL \,.\, SP}{SY} = \frac{4SP^2}{SY}$; and PO, the radius of curvature, $= \frac{2SP^2}{SY}$.

Prop. XV.

In the ellipse, to find the radius of curvature, and the chords of curvature passing through the focus, and through the centre respectively.

Let the circle be described as in the last proposition;



and let CP be a diameter, QR parallel to it, and QV parallel to the tangent PR.

By Prop. 6, $GV.VP : QV^2 :: CP^2 : CD^2$,

or since $VP = QR$, and $QV^2 = PR^2 = QR.RK$,

$$GV : RK :: CP^2 : CD^2.$$

And ultimately, when Q comes to P, GV becomes $2CP$; and RK becomes PI, the chord of curvature through the centre: hence $2CP : PI :: CP^2 : CD^2$.

$$PI = \frac{2CD^2}{CP}, \text{ the chord through the centre.}$$

Also if PM be the diameter of curvature, PM is perpendicular to the tangent at P, and therefore to CD: let it meet CD in F.

By similar triangles, $PI : PM :: PF : CP$;

whence, $PM.PF = PI.CP = 2CD^2$, by what precedes.

Hence, $PM = \frac{2CD^2}{PF}$, and $PO = \frac{CD^2}{PF}$ the radius of curvature.

Also if PL be the chord through the focus, let it meet CD in E; then $PE = AC$; for if Hh be drawn parallel to CE, $SE = Eh$; whence $2PE = PS + Ph = PS + PH$, because PH, PS make equal angles with PR, and therefore with Hh.

Join ML. Then by similar triangles, PFE, PLM,

$$PM : PL :: PE : PF;$$

hence $PL.PE = PM.PF = 2CD^2$;

$$PL = \frac{2CD^2}{PE} = \frac{2CD^2}{AC}, \text{ the chord through the focus.}$$

Cor. The radius of curvature, and the chords of curvature through the centre and focus of a hyperbola, will be expressed by the same formulæ as in the ellipse; the letters being put at corresponding points. For the properties which are employed in the demonstration are all true for the hyperbola as well as the ellipse.

Hence in the hyperbola,

$$\text{radius of curvature} = \frac{CD^2}{PF},$$

$$\text{chord through the centre} = \frac{2CD^2}{CP},$$

$$\text{chord through the focus} = \frac{2CD^2}{AC}.$$

SCHOLIUM.

DEFINITIONS FROM EUCLID, BOOK XI.

1. A *sphere* is a solid figure described by the revolution of a circle about its diameter, which remains fixed. The *axis* of the sphere is the fixed straight line about which the semicircle revolves. The *centre* of the sphere is the centre of the semicircle. A *diameter* of the sphere is any straight line which passes through the centre, and is terminated both ways by the surface of the sphere.

2. A right *cone* is a solid figure described by the revolution of a right-angled triangle about one of its sides, which remains fixed. The *axis* of the cone is the fixed straight line about which the triangle revolves. The *base* of the cone is the circle described by the side containing the right angle which revolves. The *vertex* of the cone is the extremity of the axis which is not in the base. The *slant side* of the cone is the hypothenuse of the right-angled triangle.

A *frustum* of a cone is a part cut off by a plane parallel to the base.

3. A *cylinder* is a solid figure described by the revolution of a right-angled parallelogram about one of its sides which

remains fixed. The *axis* of the cylinder is the fixed straight line. The *basis* of the cylinder are the circles described by the two sides which revolve each round a fixed extremity.

A rectangular prism or parallelopiped is equal to the product of its base and height.

PROP. XVI.

The circumferences of circles are as their radii.

For if, in two circles, any two similar parallelograms be described, the perimeters of these circles are as the radii, by the Lemma to Prop. 5, B. II. And hence this is true always, it is true at the limit, when the number of sides of the polygons is indefinitely increased. But the limit of the perimeters of each polygon is the circumference of the circumscribing circle. Therefore these circumferences are in the ratio of the radii.

COR. If π be the semi-circumference of the circle of which the diameter is 1, $2\pi r$ is the circumference of the circle of which the radius is r.

PROP. XVII.

The area of a circle is the rectangle of the circumference and half the radius.

For if any regular polygon be inscribed in the circle, the area of this polygon will be equal to the area of all the triangles of which the bases are sides of the polygon, and the vertex the centre of the circle. And each of these polygons is half the rectangle of the base, and the perpendicular drawn from the centre upon the base. Therefore the whole polygon is the rectangle of the perimeter of the polygon, and half the perpendicular drawn from the centre upon any of the sides. And this is true at the limit, when the number of sides is indefinitely increased. But at the limit the perpendicular becomes the radius of the circle, the perimeter becomes the circumference of the circle, and the area of the polygon becomes the area of the circle. Therefore the area of the circle is equal to the rectangle of the circumference and half the radius.

Cor. 1. If r be the radius, πr^2 will be the area.

Cor. 2. In like manner it may be shown, that the area of a sector is equal to the rectangle of the arc and half the radius.

Prop. XVIII.

The solid content of a cylinder is equal to the product of the base and axis.

For upon the base may be erected rectangular prisms, occupying the whole of the cylinder except those portions which are left by the rectangular form of the prisms, as if prisms and a cylinder were erected upon the figure to Prop. 2. Book II. And each of these prisms will be equal to its base multiplied by its height, that is, by the axis of the cylinder. And therefore the sum of the prisms will be equal to the sum of the bases multiplied by the axis of the cylinder. And this will be true at the limit. But, at the limit, the sum of the bases of the prisms becomes equal to the circular base of the cylinder. And the limit of the sum of the prisms becomes equal to the cylinder, by Book II. Prop. 3. Cor. 5. Therefore the cylinder is equal to the product of its base and axis.

Cor. If r be the radius of the base, a the axis, the solid content of the cylinder is $\pi r^2 a$.

Prop. XIX.

The curve surface of the cylinder is equal to the rectangle of its axis and the circumference of its base.

For if the surface be cut, along a line parallel to the axis, it may be unfolded into a rectangle of which one side is the axis of the cylinder, and the other the circumference of the base.

PROP. XX.

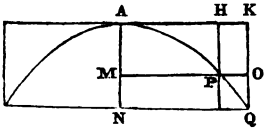
The curve surface of a (right) cone is equal to a sector of a circle of which the radius is the slant side of the cone, and the arc is the circumference of the base.

For if the curve surface of the cone be cut, along the slant side, it may be unfolded into a plane. And this plane will be a sector of a circle, because each point of the base is equally distant from the vertex; this distance being the slant side, and the arc being the circumference of the base. Hence the curve surface of the base is equal to the sector above described.

COR. If r be the radius of the base, and s the slant side, the curve surface of the cone is πrs, by Prop. 17, Cor. 2.

PROP. XXI.

The area of a parabola cut off by its ordinate is two-thirds of the circumscribing parallelogram.



Let two ordinates MP, NQ be drawn, and PH, QK parallel to the axis, meeting the tangent at the vertex; and let MP meet QK in O.

By the property of the parabola,

$$MP^2 = 4AS.AM, \quad NQ^2 = 4AS.AN, \text{ whence}$$

$$NQ^2 - MP^2 = 4AS.MN; \text{ or since } NQ = MP + PO,$$

$$2MP.PO + PO^2 = 4AS.MN; \text{ wherefore}$$

$$\begin{aligned} PO : MN &:: 4AS : 2MP + PO \\ &:: 4AS.AM : 2AM.MP + AM.PO \\ &:: MP^2 : 2AM.MP + AM.PO \\ &:: MP : 2AM + \frac{AM.PO}{MP}. \end{aligned}$$

But when Q comes to P, $\frac{AM.PO}{MP}$ vanishes. Therefore ultimately

$$PO : MN :: MP : 2AM.$$

Hence ultimately $AM.PO : MP.MN :: 1 : 2$;

or the parallelogram NP is ultimately double of the parallelogram PK.

And if a series of parallelograms be inscribed in the curvilinear spaces AMP, AHP, by taking a series of points in the curve, each parallelogram in the space AMP will ultimately be double of the corresponding parallelogram in AHP.

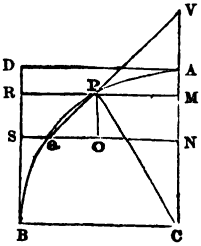
Therefore, by Book II. Prop. 4, the curvilinear space AMP is double of the curvilinear space AHP.

And these two spaces together make up the parallelogram MH. Whence AMP is two thirds and AHP one third of the parallelogram.

Prop. XXII.

If a sphere and its circumscribing cylinder have the same axis, and be cut by planes perpendicular to the axis, the intercepted portions of the surfaces are equal.

Let CA be the axis; AB a quadrant of the generating circle of the sphere; $ACBD$ a square; then the cylinder circumscribing the hemisphere is generated by the revolution of the square $ACBD$.



Let MPR, NQS be lines perpendicular to the axis, PQ the chord of the circular arc PQ. Then, when the figure revolves about the axis CA, PQ generates the surface of a frustum of a cone; and if QP meet CA in V, V will be the vertex of this cone. The surface will be the difference of the surface of the cones generated by the lines VQ, VP. It will therefore (Prop. 20) be $\pi VQ.NQ - \pi VP.MP$. And the cylindrical surface generated by the corresponding line RS will be $2\pi.BC.RS$. Hence

conical surface generated by PQ : cylindrical surf. gen. by RS

$$:: VQ.NQ - VP.MP : 2BC.RS$$

$$:: VP\left(\frac{VQ.NQ}{VP} - MP\right) : 2BC.RS$$

$$:: VP\left(\frac{NQ^2}{MP} - MP\right) : 2BC.RS; \text{ because } \frac{VQ}{VP} = \frac{NQ}{MP}$$

$$:: VP\left(2OQ + \frac{OQ^2}{MP}\right) : 2BC.RS; \text{ because } NQ = MP + OQ$$

$$:: \frac{VP}{BC}\left(2 + \frac{OQ}{MP}\right) : 2\frac{RS}{OQ}$$

$$:: 2 + \frac{OQ}{MP} : 2 \text{ ultimately;}$$

for ultimately $\frac{VP}{BC} = \frac{VP}{PC} = \frac{OP}{OQ} = \frac{RS}{OQ}$, since when Q comes up to P, the angle VPC is ultimately a right angle

$$:: 2 : 2 \text{ or } 1 : 1 \text{ ultimately, for ultimately } \frac{OQ}{MP} \text{ vanishes.}$$

Therefore the cylindrical surface generated by RS is ultimately equal to the conical surface generated by PQ. And the same is the case if other corresponding points be taken.

If therefore a polygon, inscribed in any portion of the curve AB, generate by its revolution an assemblage of conical surfaces, these surfaces will be ultimately equal, each to each, to the corresponding portions of the cylindrical surface determined by lines MR, NS.

Therefore the limit of the assemblage of conical surfaces is equal to the cylindrical surface cut off by the extreme ordinates.

But the limit of the assemblage of conical surfaces is the curve surface of the spherical zone determined by the extreme ordinates.

Therefore the surface of this spherical zone is equal to the corresponding surface of the cylinder.

Prop. XXIII.

If a parabola and its circumscribing rectangle generate solids by revolving about the axis of the parabola, the *conoid* generated by the parabola is half the circumscribing cylinder.

In the diagram to Prop. 21, let the figure revolve about the axis *AN*. Then the rectangle *NP* will generate a cylinder of which the axis is *MN*, and the base is the circle to the radius *MP*; and the rectangle *KP* will generate a hollow cylinder which is the difference of the two cylinders having the axis *AM*, and the bases the circles to the radius *AK* and *AH* respectively. Hence, Prop. 18, Cor.

$$\text{solid generated by } NP = \pi . MP^2 . MN = 4\pi . AS . AM . MN,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots \quad KP = \pi . AK^2 . AM - \pi . AH^2 . AM$$

$$= \pi . AM (AK^2 - AH^2)$$

$$= \pi . AM (NQ^2 - MP^2)$$

$$= \pi . AM (4AS . AN - 4AS . AM)$$

$$= 4\pi . AS . AM . MN.$$

Hence the solids generated by *NP* and by *KP* are equal.

And if a series of rectangles be inscribed in *AMP*, *AHP* in the same manner, the solids generated by those in the space *APH* will be equal, each to each, to the solids generated by those in *AHP*. Therefore, by Cor. to Prop. 4. Book II, the solid generated by *APM* is equal to the solid generated by *AHP*. And the two solids together make up the cylinder generated by *AMPH*. Therefore each solid is half this cylinder.

Prop. XXIV.

The solid content of a cone is one-third the solid content of the cylinder of the same base and axis.

This might be proved nearly as the last: but otherwise as follows.

Let h be the axis, k the radius of the base; and let the axis be divided into n equal parts, and, at the points of division, let lines be drawn perpendicular to the axis, and let the rectangles be completed which are circumscribed about the generating triangle of the cone. Then, when these rectangles revolve, they will generate cylinders, of which the axis in each will be $\frac{h}{n}$; and the radii of the bases of these cylinders, beginning from the vertex, will be (by similar triangles)

$$\frac{k}{n},\ \frac{2k}{n},\ \frac{3k}{n},\ \text{and so on } (n \text{ terms}).$$

Hence the sum of these cylinders will be

$$\pi\frac{h}{n}\cdot\frac{k^2}{n^2}+\pi\frac{h}{n}\cdot\frac{4k^2}{n^2}+\pi\frac{h}{n}\cdot\frac{9k^2}{n^2}+\text{\&c. } (n \text{ terms})$$

$$=\frac{\pi k^2 h}{n^3}\left\{1^2+2^2+3^2+\text{\&c. } (n \text{ terms})\right\}$$

$$=\frac{\pi k^2 h}{n^3}\left\{\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}\right\} \text{ by the summation of series}$$

(*Wood.* Alg. 419.)

$$=\frac{\pi k^2 h}{3}+\frac{\pi k^2 h}{2n}+\frac{\pi k^2 h}{6n^2}\cdot$$

And the limit of this, when n, the number of rectangles, becomes indefinitely great, is $\frac{\pi k^2 h}{3}$. But the limit of the sum of the cylinders, in that case, is the cone by B. II. Prop. 3, Cor. 6. Therefore the solid content of the cone is $\frac{1}{3}\pi k^2 h$. And that of the cylinder is $\pi k^2 h$ by Prop. 18 of this Book. Therefore, &c. Q. E. D.

BOOK IV.*

GEOMETRICAL SOLUTION OF THE DIRECT PROBLEM OF CENTRAL FORCES.

DEFINITIONS AND PRINCIPLES.

Art. 1. THE direct problem of central forces is, to find the force, tending to a given centre, which will cause a body to describe a given curve.

2. The motion is supposed to take place in non-resisting spaces.

3. *First Law of Motion.* A body in motion not acted upon by any force will go on for ever with a uniform velocity. (Mech. Euclid, B. III. Prop. 2.)

4. *Second Law of Motion.* When any force acts upon a body in motion, the motion which the force would produce in the body at rest is compounded with the previous motion of the body. (Mech. Euclid, B. III. Prop. 8.)

5. *Definition of Velocity.* Velocity is measured by the limit of the quotient of the space described divided by the time.

6. *Definition of Accelerating Force.* Uniform accelerating force is measured by the velocity which the force generates in a unit of time. (Mech. Euclid, B. III. Prop. 13.)

7. An uniform accelerating force is measured by the space through which it will urge a body from rest in a given time, according to the formula $s = \frac{1}{2}ft^2$. (Mech. Euclid, B. III. Prop. 7.)

8. Any accelerating force is *ultimately* measured by the space through which it will urge a body from rest in a given *small* time, according to the formula $s = \frac{1}{2}ft^2$.

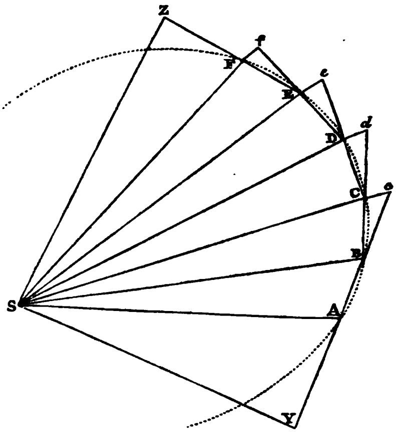
---

* Newton Principia, Book I. Section II. and III.

9. Any accelerating force will produce its effect upon a body in motion, according to the second law of motion, a small time being taken and the ultimate ratio of the effects being considered.

PROP. I.

When a body moves in a curve, acted on by forces tending to a fixed point, the areas which it describes by rays drawn to the centre of force, are in a constant plane, and are proportional to the times.



Let the time be divided into equal portions, and in the first portion let the body describe *AB*. By the first law of motion, if no force were to act on the body, it would in the second portion of time go on to *c*, in the same straight line, describing *Bc* equal to *AB*. But when the body comes to *B*, let a force tending to the centre act on it by a single instantaneous impulse, and turn the motion in the direction *BC*. Draw *cC* parallel to *BS*; and by the second law of motion, the body will describe *BC* in the second portion of time, *C* being in the plane *ASB*. Join *SC*; the triangle *SBC* is equal to *SBc**, because *cC*, *BS*, are parallel; and therefore to *SAB*, because *Bc* is equal to *BA*.

In like manner if a centripetal force towards *S* act impulsively at *C*, *D*, *E*, &c. at the end of equal successive portions of time, causing the body to describe the straight lines *CD*, *DE*, *EF*, &c.: these lines will all lie in the same plane, and the triangles *SCD*, *SDE*, *SEF* will all be equal to *SAB* and SBC. Therefore these triangles will be described in equal times, and in a constant plane. And we shall have

---

* The line *Sc* is wanting in the Figure.

$$SADS : SAFS :: \text{time in } AD : \text{time in } AF.$$

Let now the number of the portions of time in AD, AF, be augmented, and their magnitude diminished, indefinitely; and the above proportion will still be true.

Therefore also *ultimately* $SADS$: $SAFS$ :: time in AD : time in AF.

But *ultimately* the polygon $ABCDEF$, &c. becomes a curve line, (B. II. Art. 6,) and the force which acted impulsively at B, C, D, E, &c. becomes a force which acts continuously at all points*. (B. II. Art. 15.)

Therefore in this case also we have curvilinear areas $SADS$: $SAFS$ :: time in AD : time in AF.

Cor. 1. The velocity of a body attracted to a fixed centre, is inversely as the perpendicular from the fixed centre upon the tangent to the curve.

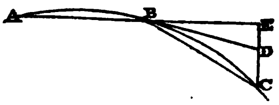
For the velocities in the polygon at two points A, D are as AB, DE, because these lines are described in equal portions of time. But if SY, SZ be perpendicular on these lines, $SY . AB = SZ . DE$, because the triangles are equal.

$$\text{Therefore vel. at } A : \text{vel. at } D :: SZ : SY.$$

And *ultimately* the velocity in the polygon becomes the velocity in the curve, and the lines AY, DZ are the tangents at A, D. Therefore, &c.

---

[*] The force in the polygon is a force which acts instantaneously at the angles, and thus produces a motion composed of straight lines. The force in the curve acts continuously. The force at each angle of the polygon is supposed to be such as will produce, instantaneously, the velocity which the continuous forces produce in the time of describing the side of the polygon. Let AB, BC be two successive sides of the polygon, $BE = BA$, and EC the space described by the action of the instantaneous force. Then if BD be a tangent to the curve ABC at B, by B. II. Prop. 13. Cor. 2, $BC = BA$ ultimately, whence $DC = DE$ ultimately, and DC is half EC ultimately. Hence, by B. II. Prop. 12. Cor. 3, DC, the deflection from the tangent is the space described by the action of the continuous force, as it ought to be.



Cor. 2. All the preceding propositions are true, when the planes in which the bodies move, and the centres of force situate in these planes, are carried with a uniform and rectilinear motion. For the relative motion of *P* with regard to *S* will be the same in both cases: and the motion of *P* will be that which will arise from the action of *S*, compounded with the motion of the plane, by Art. 4.

Prop. II.

If a body moves in a curve line in a constant plane, and by a ray drawn to a fixed point, describes areas, about that point, proportional to the times; it is urged by a central force tending to that point.

The same is true if the point, instead of being fixed, be moving uniformly in a straight line.

Case 1. When the point is fixed. Every body which moves in a curve, is deflected from a straight line by some force acting upon it. If the body were to describe the polygon *ABCDEF*, (see Fig. p. 74,) describing the equal triangles *SAB*, *SBC*, &c. in equal times, it must at *B* be acted on by a force parallel to *cC* (*Bc* being equal to *AB*) by the second law of motion. And *cC* is parallel to *BS*, because *SBC* is equal to *SAB* or *SBC*: therefore the force in the polygon acts to the centre *S*. But *ultimately* the motion in the polygon will coincide with the motion in the curve, and the force in the polygon will be proportional to the force in the curve. Therefore in the curvilinear motion the proposition is true.

Case 2. And by Cor. 2 to Prop. 1, the force is the same, whether the plane in which the motion takes place be at rest, or move, along with the body, the figure described, and the centre of force, uniformly in a right line.

Cor. 1. In a non-resisting space, if the area is not proportional to the time, the force does not tend to the point to which the rays are drawn. It deviates to the side towards which the motion is, if the description of areas is accelerated; to the opposite side if the description is retarded.

Cor. 2. In a resisting medium also, if the description of areas is accelerated, the direction of the force deviates to the side towards which the body moves.

Prop. III.

If a body revolve about another body which is moving in any manner whatever, and if the first body describe, about the second, areas proportional to the times; the first body is acted on by a centripetal force tending to the second body, and also by the whole of the force by which the second body is acted on.

Let *L* be the first, *T* the second body. Let *F* be the force by which *T* is acted on; and let a new force equal to *F*, and in the opposite direction, act upon *T* and *L* in parallel lines. Then the motion of *L* with respect to *T*, will be exactly the same as before, by Cor. 2, Prop. 1. But on this supposition *T* is acted upon by two equal and opposite forces; and is therefore in the same condition as if it were acted upon by no force; and therefore will either remain at rest, or move uniformly in a straight line. Hence by Prop. 2, the body *L* is on this supposition acted upon simply by a force tending to the center *T*. But if we now suppose *L* to be acted upon by the force *F*, in a direction parallel to that in which it really acts on *T*, *L* will be reduced to its real condition. Therefore *L* is really acted on by a force tending to *T*, and by the force *F* which acts on *T*.

Cor. 1. Hence if *L* describe areas about *T*, proportional to the times, and if from the whole force (simple or compound) which acts on *L*, we subtract the force which acts on *T*, the residue of force which acts on *L*, will tend to *T*.

Cor. 2. And if the areas which *L* describes be very nearly proportional to the times, the residue of the force so obtained will tend to *T* very nearly.

Cor. 3. And conversely, if the residue of the force so obtained tend very nearly to *T*, the areas will be very nearly proportional to the times.

Cor. 4. If L describe about T areas which are very far from being proportional to the times, L is either acted upon by no centripetal force tending to T, or this force is mixed and compounded with very powerful actions of other forces, besides the force which act on T. And the whole residual force on L tends to some other centre moveable or immoveable.

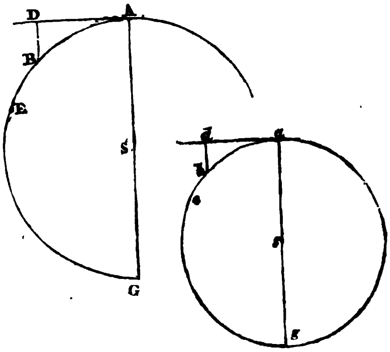
Since it appears that the equable description of areas about a centre, by a moving body, indicates necessarily that the main force by which the body is affected, deflected from a straight line, and retained in its orbit, tends to that centre, if the body be governed by the ascertained laws of motion; we may in any case, as for instance in the cases of the earth, moon, planets, and satellites, suppose that those centres about which the heavenly bodies describe equal areas, are the centres about which the motions of such bodies are performed in free space, according to the laws of motion.

Prop. IV.

When bodies describe different circles with uniform motions, the forces tend to the centres of the circles, and are as the squares of arcs described in equal times, divided by the radii of the circles.

The forces tend to the centres of the circles by Prop. 2.

Let AB, ab be arcs described in equal times, BD, bd perpendicular to the tangents at A, a. Then DB, db are the spaces through which the bodies are deflected from the tangents, by the action of the forces to S and s; and *ultimately* are as the forces (Arts. 6 and 7). Now ultimately,



$$DB = \frac{AB^2}{2AS}, \quad db = \frac{ab^2}{2as}.$$

Therefore,

$$\text{force at } A : \text{force at } a :: \frac{AB^2}{2AS} : \frac{ab^2}{2as} \text{ ultimately.}$$

But if AE, ae be arcs described in any other equal times, $AE : AB :: ae : ab$, because the motions are uniform.

$$\text{Therefore } AB : ab :: AE : ae : \text{and}$$

$$\text{Force at } A : \text{force at } a :: \frac{AE^2}{2AS} : \frac{ae^2}{2as} :: \frac{AE^2}{AS} : \frac{ae^2}{as}.$$

Cor. 1. The arcs AE, ae are as the velocities: hence in circles if F be the force, V the velocity, R the radius, $F \propto \frac{V^2}{R}$.

Cor. 2. Also if P be the periodic time, (or time of describing the whole circle)

$$P \text{ is as } \frac{\text{circumference}}{\text{velocity}}, \text{ and therefore as } \frac{R}{V}.$$

$$\text{Therefore } F \propto R \cdot \frac{V^2}{R^2} \propto \frac{R}{P^2}.$$

Cor. 3. If P be constant, $V \propto R$, $F \propto R$.

Cor. 4. If $P \propto \sqrt{R}$, $V \propto \sqrt{R}$, $F \propto 1$, and the force is the same in all circles.

Cor. 5. If $P \propto R$, $V \propto 1$, $F \propto \frac{1}{R}$.

Cor. 6. If $P \propto R^{\frac{3}{2}}$, $V \propto \frac{1}{\sqrt{R}}$, $F \propto \frac{1}{R^2}$.

Cor. 7. If $P \propto R^n$, $V \propto \frac{1}{R^{n-1}}$, $F \propto \frac{1}{R^{2n-1}}$.

Cor. 8. What has been proved of circles, is true of similar portions of similar curves having centres similarly situated. Thus in two such points of two similar curves, force in one curve (F) : force in the other (f) :: $DB : db :: \frac{AB^2}{AG} : \frac{ab^2}{ag}$.

And $AB : ab :: V : v$, these being the velocities. Also, since the curves are similar and AG, ag are the diameters of curvature, $AG : ag :: AS : as :: R : r$.

$$\text{Hence } F : f :: \frac{V^2}{R} : \frac{v^2}{r}.$$

Cor. 9. Let V be the velocity in a circle at A, f the force, t the time of describing AB, R the radius,

$$AB = Vt,\ BD = \tfrac{1}{2}ft^2. \text{ (Art. 8.) Hence } \frac{V^2t^2}{\frac{1}{2}ft^2} = \frac{AB^2}{BD}.$$

$$\text{But ultimately } \frac{AB^2}{BD} = 2R; \text{ hence } \frac{V^2}{f} = R;\ f = \frac{V^2}{R}.$$

Cor. 10. Let P be the periodic time: π the ratio of the circumference of a circle to the diameter. Then $2\pi R = PV$,

$$4\pi^2R^2 = P^2V^2 \text{ and } F = \frac{V^2}{R} = \frac{4\pi^2R}{P^2}.$$

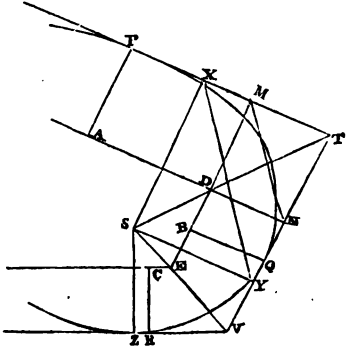
Cor. 11. In any time t the arc described is Vt; the space fallen through by force f is $\frac{1}{2}ft^2$. Now $V^2t^2 = \frac{1}{2}ft^2 . 2R$, because $R . f = V^2$. Therefore the arc described in t is a mean proportional between the space fallen through in t and the diameter.

Lemma to Prop. V.

If $TXSY$ be any quadrilateral figure, and if DM, DN be parallel to SX, SY, and

$$DM : DN :: SX : SY;$$

T, D, S shall be in a straight line.



For because SX, SY are parallel to DM, DN, the angle MDN is equal to XSY. Also the sides are proportional by supposition: therefore the triangles XSY, MDN are similar; and the angle SXY

$= DMN$. But the angle $SXT = DMT$; therefore $YXT = NMT$, and the triangles YXT, NMT are similar.

Therefore $YX : XT :: NM : MT$;
also $SX : YX :: DM : NM$;
therefore $SX : XT :: DM : MT$;

and the triangles SXT, DMT are similar, and the angle $DTM = STX$, whence D is in the straight line ST.

Prop. V.

Having given the velocity and direction of motion at three points of a figure described by a body acted on by a central force; to find the centre of force.

Let PT, QV, RZ (see fig. p. 80), be the three directions or tangents at P, Q, R, meeting in T and V. At P, Q, R erect three perpendiculars PA, QB, RC, reciprocally proportional to the velocities; that is, such that

$PA : QB ::$ vel. at Q : vel. at P,
$QB : RC ::$ vel. at R : vel. at Q.

Draw AD, DBE, EC, parallel to PT, TV, VR, and meeting in D, E; join TD, VE; these lines produced to meet will give the centre of force.

For let S be the centre, SX, SY, and SZ perpendiculars upon the tangents: DM, DN parallel to SX, SY. By Prop. 1. Cor. 1, we have

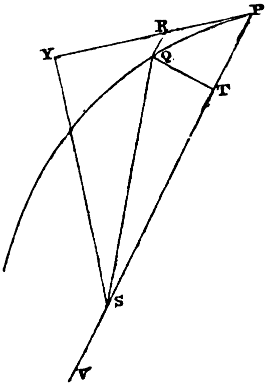
$SX : SY ::$ vel. at Q : vel. at P $:: PA : QB :: DM : DN$.

Hence, by the Lemma, SDT is a straight line. For a like reason SEV is a straight line. Therefore S is at the concourse of the two lines TD, VE.

Prop. VI.

If a body revolve about a fixed centre of force, and if, at the extremity of a small arc, the subtense of the angle of contact at any point be drawn parallel to the line joining the centre of force and that point;

The force tending to the centre is ultimately as the subtense directly, and as the square of the time of describing the arc inversely, when the time is indefinitely diminished.



Let a body revolve in the line *PQ*, about the centre *S*: and *PR* being a tangent, let *QR* be parallel to *SP*.

Let *F* be the force at *P*, *T* the time of describing *PQ*. Then by Art. 8, *RQ* is ultimately equal to the space described in *T* by *F* continued constant; that is, to $\frac{1}{2}FT^2$. Therefore

$$QR = \tfrac{1}{2}FT^2 \textit{ ultimately},$$

$$\text{and } F = \frac{2QR}{T^2} \textit{ ultimately}.$$

Whence this proposition is manifest.

Cor. 1. Let *QT* be drawn perpendicular on *SP*: then (B. II. Prop. 8) the area *SPQ* is *ultimately* equal to $\frac{1}{2}SP \,.\, QT$. But if *A* be the area described in a unit of time, by Prop. 1.

Area $SPQ : A :: T : 1$; hence $\frac{1}{2}SP \,.\, QT = A \,.\, T$ *ultimately*,

$$\text{and } \tfrac{1}{4}SP^2 \,.\, QT^2 = A^2 \,.\, T^2 \textit{ ultimately}.$$

Multiply this into the equation in the proposition, and we have $\frac{1}{4}SP^2 \,.\, QT^2 \,.\, F = 2A^2 \,.\, QR$ *ultimately*,

$$\text{whence } F = \frac{8A^2 \,.\, QR}{SP^2 \,.\, QT^2} \textit{ ultimately}.$$

Cor. 2. Let *SY* be drawn perpendicular on the tangent *PR*: then (B. II. Prop. 7 and 8) the area *SPQ* is *ultimately* equal to $\frac{1}{2}SY \,.\, PQ$: hence by the same reasoning as in the last corollary, we shall have

$$F = \frac{8A^2 \,.\, QR}{SY^2 \,.\, PQ^2} \textit{ ultimately}.$$

Cor. 3. Let PV be the chord of the circle of curvature passing through S: then, by B. II. Prop. 13, Cor. 1, $PV.QR = PQ^2$, and if we put $PV.QR$ for PQ^2 in Cor. 2, we find

$$F = \frac{8A^2}{SY^2 . PV}.$$

Cor. 4. If V be the velocity of the body at P, $PQ = V.T$ *ultimately* by Art. 5 of this Book; and $PQ^2 = V^2T^2$; therefore $PV.QR = V^2T^2$ *ultimately*: but $QR = \frac{1}{2}FT^2$ *ultimately*, as in the proposition: hence $\frac{1}{2}PV.F = V^2$, and

$$F = \frac{V^2}{\frac{1}{2}PV}.$$

Cor. 5. If F be a constant force and V the velocity acquired in falling through a space S, $V^2 = 2FS$, by Cor. 5 to B. II. Prop. 12. Now $V^2 = 2F.\frac{1}{4}PV$, by last Cor. Hence $S = PV$; or the velocity in any curve is equal to that acquired in falling down $\frac{1}{4}$ the chord of curvature passing through the centre of force, the force being constant during this fall.

It appears from this Proposition and its corollaries, that a curve being given, which is described by the action of a central force, the law and magnitude of the force may be determined by determining the *ultimate* value of any of the following quantities:

$$\frac{2QR}{T^2},\quad \frac{8A^2.QR}{SP^2.QT^2},\quad \frac{8A^2.QR}{SY^2.PQ^2};$$

or the value of either of the following quantities

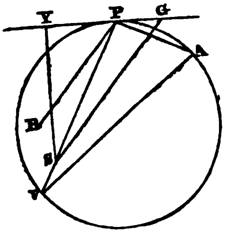
$$\frac{8A^2}{SY^2.PV},\quad \frac{V^2}{\frac{1}{2}PV}.$$

The determination of the law and magnitude of the force, when the curve is given, is the solution of *the direct Problem of central forces*.

The following Propositions are Examples of such solutions.

PROP. VII.

A body revolves in the circumference of a circle, the force tending to any point: to find the force (F).



Let S be the centre of force, and SY perpendicular on the tangent PY. Then by Cor. 3 to last Prop. $F = \dfrac{8A^2}{SY^2 . PV}$.

Let AV be the diameter: then by similar triangles AVP, PSY,

$$AV : PV :: SP : SY = \frac{PV . SP}{AV};$$

putting this value for SY, $F = \dfrac{8A^2 . AV^2}{SP^2 . PV^3}$.

Hence in different points of the curve the force varies inversely as $SP^2 . PV^3$.

COR. 1. Hence if S the centre of force be in the circumference of the circle, the points S and V coincide, and

$$F = \frac{8A^2 . AS^2}{SP^5}.$$

The force in this case varies inversely as the fifth power of the distance.

COR. 2. Let the force $= \dfrac{\mu}{SP^5}$, μ being constant. Also let P be the periodic time: then $P . A =$ the whole area of the circle $= \dfrac{\pi . AS^2}{4}$. Hence $A = \dfrac{\pi . AS^2}{4P}$.

$$\text{And } \mu = (8A^2 . AS^2 =) \frac{\pi^2 . AS^6}{2P^2},$$

$$\text{whence } P^2 = \frac{\pi^2}{2\mu} AS^6.$$

About the same centre, μ is the same, and $P \propto AS^3$.

Cor. 3. By Cor. 1. Prop. 1.

$$\frac{\text{vel. at } P}{\text{vel. at } A} = \frac{SA}{SY} = \frac{SA}{SP} \cdot \frac{SP}{SY} = \frac{SA^2}{SP^2} \text{ by similar triangles.}$$

Cor. 4. Let R, S be two centres of force, A the area described in a unit of time, when the body moves in a circle about R; B the area described in a unit of time when the body moves in the same circle about S; F, G, the two forces; PT* the chord through R;

$$\text{Then, } F = \frac{8A^2 . AV}{RP^2 . PT^3},$$

$$G = \frac{8B^2 . AV}{SP^2 . PV^3}.$$

Hence,

$$F : G :: \frac{A^2}{RP^2 . PT^3} : \frac{B^2}{SP^2 . PV^3}, \text{ or } :: A^2 \frac{PV^3}{PT^3} : B^2 . \frac{RP^2}{SP^2}.$$

Let SG parallel to TP meet the tangent: then by similar triangles PSG, TPV, $PV : PT :: SG : SP$;

$$\text{hence, } \frac{PV}{PT} = \frac{SG}{SP},$$

$$\text{And } F : G :: A^2 . \frac{SG^3}{SP^3} : B^2 \frac{RP^2}{SP^2}, \text{ or } :: A^2 . SG^3 : B^2 . SP . RP^2.$$

Cor. 5. In any curve touching GP at P, and having PVA for its circle of curvature, the subtenses of the angles of contact are *ultimately* equal to those of the corresponding arcs of the circle; and hence (Prop. 6.) the forces in such a curve are ultimately equal to the forces in the circle in the same directions, and for the same velocity. Hence in any curve, if F be the force tending to R, G the force tending to S, SG parallel to RP, A and B the areas in a unit of time described about R and S respectively,

$$F : G :: A^2 . SG^3 :: B^2 . SP . RP^2.$$

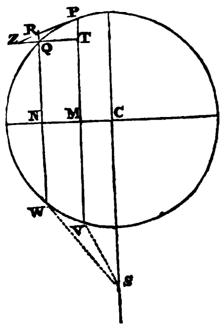
---

* PT, TV are wanting in the figure.

PROP. VIII.

Let a body move in a semicircle *PQA*, the force tending in parallel lines *PM*, *QN*: to find the law and magnitude of the force.

This may be considered as a case of a body acted on by a central force, by conceiving the centre to be indefinitely distant, so that all lines tending to it may be assumed to be parallel.



Let C be the centre of the circle, and CMN a diameter perpendicular to the direction of the force. On account of the similar triangles CPM, PZT, RZQ, we have

$$CP^2 : PM^2 :: PR^2 : QT^2;$$

Also by the property of the circle $PR^2 = QR . RW$, and *ultimately*

$$PR^2 = QR . PV = QR . 2PM.$$

Hence

$$CP^2 : PM^2 :: QR . 2PM : QT^2;$$

$$CP^2 . QT^2 = QR . 2PM^3;$$

$$\frac{QR}{QT^2} = \frac{CP^2}{2PM^3} \text{ ultimately.}$$

Also $\frac{A^2}{SP^2}$ is constant; for A is constant, and $SP = SC$ *ultimately*, when S is indefinitely distant, and is therefore constant.

$$\text{Now, } \frac{8A^2}{SP^2} = \frac{A^2 . 2MN^2}{\frac{1}{4} SP^2 . QT^2} = \frac{2MN^2}{T^2},$$

because $\frac{1}{2} SP . QT : A :: T : 1$, (Prop. 1.)

Hence, $\frac{MN}{T}$ is constant: let $\frac{MN}{T} = U$, and multiplying together the expressions for

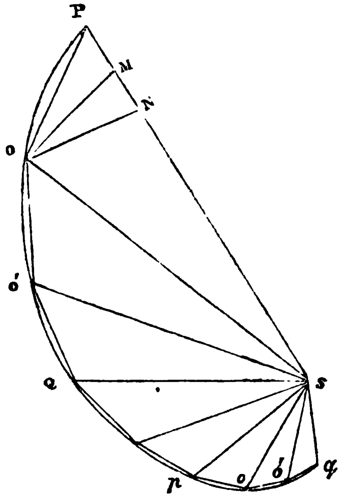
$\frac{QR}{QT^2}$ and for $\frac{8A^2}{SP^2}$, we have the force at $P = \frac{CP^2 . U^2}{PM^3}$.

Here U is the velocity parallel to CA.

Hence it appears that the force is inversely as PM^3.

LEMMA TO PROP. IX.

In the spiral which cuts all its rays at equal angles, rays at equal angular distances have everywhere equal ratios.



Let PSO, OSO', $O'SQ$... pSo, oSo', $o'Sq$, be equal angles; and let the angles SPO, SOO', $SO'Q$...Spo, Soo', $So'q$, be all equal. Then when SPO, OSO', &c. are indefinitely increased in number and diminished in magnitude, the limit of $POO'Q$.... $poo'q$ is the equiangular spiral.

The triangles PSO, OSO, $O'SQ$, &c., pSo, oSo', $o'Sq$, &c. are all similar; hence, whatever be the number of angles, we have such proportions as these,

$$SP : SO :: Sp : So;$$

$$SO : SO' :: So : So';$$

$$SO' : SQ :: So' : Sq; \text{ therefore}$$

$$SP : SQ :: Sp : Sq;$$

and therefore in the curve which is the limit of the polygon

$$SP : SQ :: Sp : Sq.$$

COR. 1. Let β be the angle which the curve makes with the ray: and let the angle QSP be $=\gamma$, and the ratio $SQ : SP$ be $1 : 1+c$. Let the angle QSP be divided into equal parts, PSO, &c. each $=\frac{\gamma}{n}$; therefore the ratios $SO : SP$, &c. are all equal; and $SO^n : SP^n :: SQ : SP$; or

$$SO : SP :: SQ^{\frac{1}{n}} : SP^{\frac{1}{n}} :: 1 : (1+c)^{\frac{1}{n}}.$$

Now if OM be an arc with centre S, meeting SP, and ON a perpendicular on SP, we have

$$ON = SO \,.\, \sin PSO, \; NP = MP \textit{ ultimately}, = SP - SO;$$

$$\text{hence, } \tan SPO = \frac{ON}{NP} = \frac{SO \sin PSO}{SP - SO}, \text{ ultimately};$$

$$= \frac{\sin PSO}{\dfrac{SP}{SO} - 1};$$

whence, β being the limit of SPO,

$$\tan \beta = \frac{\sin \dfrac{\gamma}{n}}{(1+c)^{\frac{1}{n}} - 1}, \; n \text{ being infinite};$$

$$= \frac{\gamma}{n(1+c)^{\frac{1}{n}} - n}.$$

Now, by the binomial theorem,

$$n(1+c)^{\frac{1}{n}} - n = n\left\{1 + \frac{1}{n}c + \frac{\frac{1}{n}\cdot\left(\frac{1}{n} - 1\right)}{1\,.\,2}c^2 + \&c.\right\} - n$$

$$= c + \frac{\frac{1}{n} - 1}{1\,.\,2}c^2 + \frac{\left(\frac{1}{n} - 1\right)\left(\frac{1}{n} - 2\right)}{1\,.\,2\,.\,3}c^3 + \&c.$$

and since n is infinite, this is $= c - \dfrac{c^2}{2} + \dfrac{c^3}{3} - \&c.$

$$\text{Hence } \tan \beta = \frac{\gamma}{c - \dfrac{c^2}{2} + \dfrac{c^3}{3} - \&c.} = \frac{\gamma}{\mathrm{l}(1+c)}.$$

COR. 2. Hence $\mathrm{l}(1+c) = \dfrac{\gamma}{\tan \beta}$; and

$1 + c = e^{\frac{\gamma}{\tan \beta}}$, e being the base of natural logarithms.

COR. 3. If PSO, OSO', &c. be taken all equal, the similar triangles thus formed will be in geometrical progression for

$$\frac{OSO'}{PSO} = \frac{SO.SO'}{SP.SO} = \frac{SO^2}{SP^2}.$$

Hence the whole curvilinear area will be the sum of this infinite decreasing geometrical progression *ultimately*, PSO being diminished without limit, (Book II. Prop. 3,) and this sum gives

$$\text{area} = \frac{\text{triangle } PSO}{1 - \dfrac{SO^2}{SP^2}}.$$

Now if the angle $PSO = \gamma$, and $\dfrac{SO}{SP} = \dfrac{1}{1+c}$,

$$\text{triangle } PSO = \tfrac{1}{2}\, SO.SP \sin\gamma,$$

$$\text{hence area} = \frac{\frac{1}{2} SO.SP\sin\gamma}{1 - \dfrac{SO^2}{SP^2}} = \frac{1}{2}\,\frac{SP^2\dfrac{SP}{SO}\sin\gamma}{\dfrac{SP^2}{SO^2} - 1}$$

$$= \frac{SP^2}{2}\,\frac{(1+c)\sin\gamma}{(1+c)^2 - 1}.$$

But, by Cor. 2, $\sin\gamma = \gamma$ *ultimately* $= (c - \dfrac{c^2}{2} + \dfrac{c^3}{3} - \text{\&c.}) \tan\beta$,

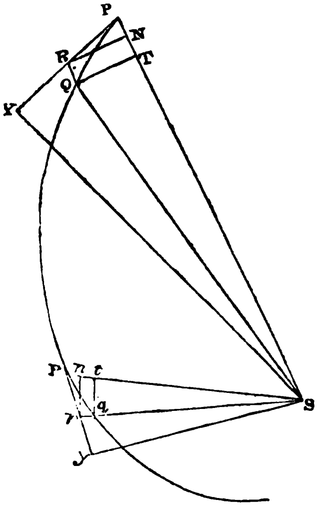
hence

$$\text{area} = \frac{SP^2}{2}\,\frac{(1+c)(c - \text{\&c.})}{2c + c^2}\tan\beta = \frac{SP^2}{2}\,\frac{(1+c)(1 - \text{\&c.})}{2+c}\tan\beta,$$

and *ultimately*, since $\gamma = 0$ and $c = 0$, this area $= \dfrac{SP^2}{4}\tan\beta$.

PROP. IX.

Let the body revolve in the equiangular spiral just described: to find the force.



Let PSQ, pSq be two small equal angles; and since the angles SPR, Spr are also equal, and their supplements PRQ, prq; and also

$$SP : SQ :: Sp : Sq$$

by the Lemma to this proposition; the figures $SPRQ$, $Sprq$ are similar. And QT will be a similar line in each figure; hence

$$\frac{QR}{QT} = \frac{qr}{qt} \text{ and}$$

$$\frac{QR}{QT^2} : \frac{qr}{qt^2} :: \frac{1}{QT} : \frac{1}{qt} :: \frac{1}{SP} : \frac{1}{sp}.$$

Now if the angle PSQ be different from pSq, $\frac{QR}{QT^2}$ remains the same as before, by B. II. Prop. 13. Hence, in all cases,

$$\frac{QR}{QT^2} : \frac{qr}{qt^2} :: \frac{1}{SP} : \frac{1}{Sp};$$

$$\text{and } \frac{QR}{SP^2 . QT^2} : \frac{qr}{Sp^2 . qt^2} :: \frac{1}{SP^3} : \frac{1}{Sp^3}:$$

and the force is inversely as the cube of the distance.

COR. 1. To find the actual value of the force: let

$$PSQ = \gamma, \ \frac{SP}{SQ} = 1 + c.$$

$$QR = TN = PT - PN = SP - SQ \cos PSQ - RN \operatorname{cotan} RPN$$
$$= SQ (1 + c - \cos\gamma - \sin\gamma \cot\beta)$$

$QT^2 = SQ^2 \sin^2 PSQ = SQ^2 \sin^2\gamma$, and $SQ = SP$ ultimately;

$$\text{hence } \frac{QR}{QT^2} = \frac{1 + c - \cos\gamma - \sin\gamma \cot\beta}{SP \sin^2\gamma}.$$

Now when γ is very small, by Cor. 2,

$$1 + c = e^{\frac{\gamma}{\tan\beta}} = 1 + \frac{\gamma}{\tan\beta} + \frac{\gamma^2}{1 \,.\, 2 \,.\, \tan^2\beta} + \&c.$$

$$\cos\gamma = 1 - \frac{\gamma^2}{1 \,.\, 2} + \&c.$$

$$\sin\gamma \cot\beta = \frac{\gamma}{\tan\beta} + \&c.$$

$$\sin^2\gamma = \gamma^2 - \&c.$$

hence, neglecting terms beyond γ^2, because the ultimate value is wanted,

$$\frac{QR}{QT^2} = \frac{\frac{\gamma^2}{2\tan^2\beta} + \frac{\gamma^2}{2}}{SP \,.\, \gamma^2} = \frac{1}{2SP}\left(\frac{\cos^2\beta}{\sin^2\beta} + 1\right) = \frac{1}{2SP\sin^2\beta}.$$

$$\text{Hence force} = \frac{8A^2\,QR}{SP^2\,QT^2} = \frac{4A^2}{\sin^2\beta \,.\, SP^3}.$$

COR. 2. The figure, taken to any distance from the centre, is always similar to itself. Lemma to Prop. 9. Hence (B. II. Prop. 5) the area is as SP^2; and the area A being constant, the time of descending to the centre from P is as SP^2.

COR. 3. By Lemma to Prop. 9, whole area $= \frac{SP^2}{4} . \tan\beta$.

$$\text{Hence time to centre} = \frac{SP^2}{4A}\tan\beta.$$

Let force $= \frac{\mu}{SP^3}$; hence by Cor. 1, $\mu = \frac{4A^2}{\sin^2\beta}$, $A = \frac{\mu^{\frac{1}{2}}\sin\beta}{2}$.

$$\text{Whence time to centre} = \frac{SP^2}{2\mu^{\frac{1}{2}}\cos\beta}.$$

COR. 4. By Cor. 1. to Prop. 1.

$$\frac{\text{vel. at } P}{\text{vel. at } p} = \frac{Sy}{SY} = \frac{Sp}{SP}, \text{ by similar triangles.}$$

Prop. X.

A body revolves in an ellipse, the centre of force being in the centre of the ellipse: to find the force.

Let CA, CB (see diagram to Prop. 11, QR and QT being properly altered) be the semiaxis major and minor, P any point, and DK a diameter conjugate to the diameter PG; QR parallel, QT perpendicular to CP; Qv parallel to the tangent PR; PF perpendicular to DK.

By Conics (B. III. Prop. 6.)

$$Pv \,.\, vG : Qv^2 :: CP^2 : CD^2.$$

also $$Qv^2 : QT^2 :: CP^2 : PF^2 \text{ by similar triangles.}$$

Hence $$Pv \,.\, vG : QT^2 :: CP^4 : CD^2 \,.\, PF^2$$

$$:: CP^4 : AC^2 \,.\, BC^2, \text{ B. III. Prop. 7.}$$

Therefore $$\frac{Pv \,.\, vG}{QT^2} = \frac{CP^4}{AC^2 \,.\, BC}.$$

Put QR for its equal Pv; and for vG, put $2CP$, to which it is *ultimately* equal; and dividing,

$$\frac{QR}{CP^2 \,.\, QT^2} = \frac{CP}{2AC^2 \,.\, BC^2};$$

whence we have (by Prop. 6. Cor. 3, SP being here CP).

$$\text{Force} = \frac{4A^2 \,.\, CP}{AC^2 \,.\, BC^2}.$$

Hence the force is as the distance CP.

Cor. Let the force $= \mu \,.\, CP$; therefore

$$\frac{4A^2}{AC^2 \,.\, BC^2} = \mu, \quad A = \mu^{\frac{1}{2}} \frac{AC \,.\, BC}{2}.$$

And the periodical time $$= \frac{\text{whole area}}{A} = \frac{\pi \,.\, AC \,.\, BC}{A}$$

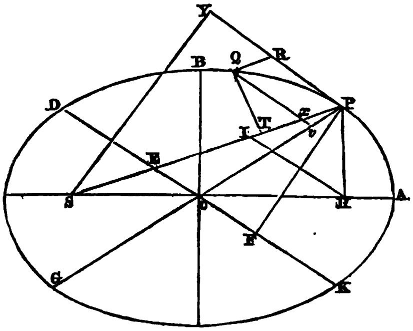
$$= \frac{2\pi}{\mu^{\frac{1}{2}}}.$$

Hence the periodic time is the same, about the same centre, whatever be the magnitude of the major axis of the ellipse.

PROP. XI.

A body revolves in an ellipse; it is required to find the force tending to the focus of the ellipse.

Let S be the focus, PQ a small arc, QR the subtense parallel to SP, QT perpendicular to SP; C the centre, PC a diameter, Qv an ordinate parallel to the tangent at P, cutting SP in x.



Let CD, conjugate to CP, meet SP in E; PE is equal to the semiaxis major AC; for if H be the other focus, and HI parallel to CE, we have $SC : CH :: SE : EI$; therefore $EI = ES$ and

$$2PE = PE + ES + PE - EI = PS + PI = PS + PH,$$

because IH being parallel to the tangent at P, makes equal angles with PS, PH; (B. III. Prop. 3.) But $PS + PH = 2AC$. Hence $PE = AC$. Now we have

$$QR\ (= Px) : Pv :: PE : PC \text{ by similar triangles.}$$

$$\begin{array}{rcccccc} \text{Therefore} & Gv.QR & : & Gv.vP & :: & PE.PC & : & PC^2, \\ \text{by Conics,} & Gv.vP & : & Qv^2 & :: & PC^2 & : & CD^2, \\ \text{ultimately} & Qv^2 & : & Qx^2 & :: & 1 & : & 1; \\ \text{and by sim. tri.} & Qx^2 & : & QT^2 & :: & PE^2 & : & PF^2. \end{array}$$

Compounding these four proportions, and observing that ultimately $Gv = 2PC$, we have *ultimately*,

$$2PC.QR : QT^2 :: PE^3.PC : CD^2.PF^2$$

$$:: AC^3.PC : AC^2.BC^2.$$

$$\text{Hence } \frac{2PC.QR}{QT^2} = \frac{AC.PC}{BC^2},\ \frac{QR}{QT^2} = \frac{AC}{2BC^2}.$$

Now $\frac{2BC^2}{AC} = L$ the *latus rectum*.

Hence $\frac{QR}{QT^2} = \frac{1}{L}$.

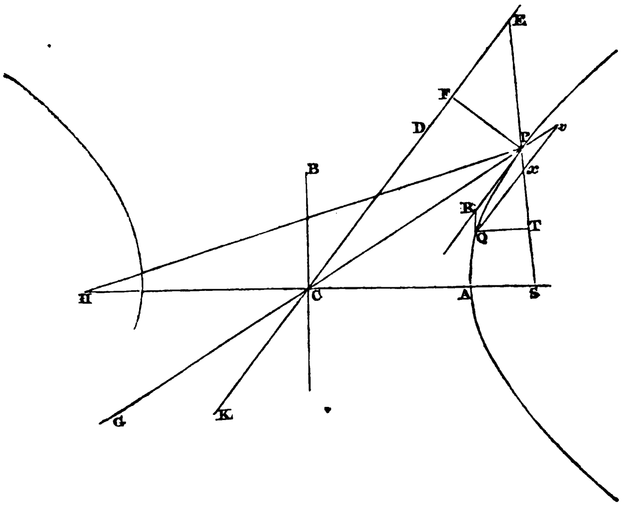
The force $= \frac{8A^2 QR}{QT^2 . SP^2}$ is $= \frac{8A^2}{L . SP^2}$.

Hence the force varies inversely as SP^2.

Prop. XII.

A body moves in a hyperbola: it is required to find the force tending to the focus.

As before, let CA, CB be the semiaxis major and minor, PG, DK two conjugate diameters; S the focus, Qxv an ordinate, parallel to the tangent at P and meeting SP in x; QR the subtense parallel to SP.



Let SP meet CD in E; PE is equal to the semiaxis-major; for if H be the other focus, and if, in the figure, HI

were drawn parallel to CE, meeting SP produced in the point I, we should have $SC : CH :: SE : EI$. Therefore $EI = ES$, and $2PE = EI + EP - (ES - EP) = IP - SP = HP - SP$, (because HI being parallel to the tangent at P, makes equal angles with PI, PH). But $HP - SP = 2AC$. Hence $PE = AC$. And we have

$QR\,(= Px) : Pv :: PE : PC$, by similar triangles.

Therefore $Gv \,.\, QR : Gv \,.\, vP :: PE \,.\, PC : PC^2$,

by Conics, $Gv \,.\, vP : Qv^2 :: PC^2 : CD^2$,

ultimately, $Qv^2 : Qx^2 :: 1 : 1$;

and by sim. tri. $Qx^2 : QT^2 :: PE^2 : PF^2$.

Compounding, we have *ultimately* (Gv being then $= 2PC$),

$$2PC \,.\, QR : QT^2 :: PE^3 \,.\, PC : CD^2 \,.\, PF^2$$

$$:: AC^3 \,.\, PC : AC^2 \,.\, BC^2, \text{ by Conics.}$$

Whence $\dfrac{QR}{QT^2} = \dfrac{AC}{2BC^2} = \dfrac{1}{L}$; L being the *latus rectum*.

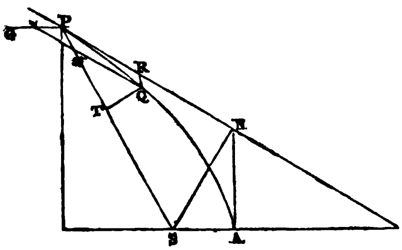
$$\text{Hence the force} = \frac{8A^2 \,.\, QR}{QT^2 \,.\, SP^2} = \frac{8A^2}{L \,.\, SP^2};$$

and the force varies inversely as the square of the distance.

In a similar manner it may be shewn, that if the body move in the opposite hyperbola, it must be acted on by forces tending *from*, instead of *to*, the centre, and varying inversely as the square of the distance.

PROP. XIII.

Let a body move in a parabola: it is required to find the force tending to the focus of the figure.



Let S be the focus, A the vertex, P any point, QR the subtense parallel to SP, QT perpendicular to SP, PG parallel to the axis, Qxv parallel to the tangent; SN perpendicular to the tangent.

Then PG, PS make equal angles with the tangent, (B. III. Prop. 1,) to which Qxv is parallel: therefore xPv is an isosceles triangle; and $Pv = Px = QR$. Now, by Conics (B. III. Prop. 3), $4SP.Pv = Qv^2$; or $4SP.QR = Qx^2$ ultimately. (B. II. Prop. 6.)

But $Qx^2 : QT^2 :: SP^2 : SN^2 :: SP : SA$; (B. III. Prop. 2.)

$$\text{whence ultimately } 4SP.QR : QT^2 :: SP : SA.$$

$$\text{And } \frac{QR}{QT^2} = \frac{1}{4SA}.$$

$$\text{Hence the force} = \frac{8A^2.QR}{SP^2.QT^2} = \frac{2A^2}{SA.SP^2}.$$

PROP. XIV.

When several bodies revolve in ellipses about the same centre of force, varying inversely as the square of the distance, the latus rectum in each, is as the square of the area described by its ray in a unit of time.

By Prop. 11, the force in this case is $\frac{8A}{L.SP^2}$, where A is the area described in a unit of time, and L the latus rectum. But about the same centre, the force is $\frac{\mu}{SP^2}$; therefore

$$\mu = \frac{8A^2}{L}, \quad L = \frac{8A^2}{\mu}, \text{ or } L \text{ is as } A^2.$$

COR. Hence $A = \frac{\sqrt{\mu} \cdot \sqrt{L}}{2\sqrt{2}}$.

PROP. XV.

On the same suppositions, the periodic times in the ellipses, are in the sesquiplicate ratio of the major axis (that is, in the ratio of the power of which the index is $\frac{3}{2}$).

For the whole area $= \pi \cdot AC \cdot BC$. Hence

$$\text{periodic time} = \frac{\text{whole area}}{\text{area in time } 1},$$

$$= \frac{\pi AC \cdot BC}{\frac{\sqrt{\mu} \cdot \sqrt{L}}{2\sqrt{2}}} = \frac{2\pi AC \cdot BC}{\sqrt{\mu}\frac{BC}{\sqrt{AC}}} = \frac{2\pi AC^{\frac{3}{2}}}{\mu^{\frac{1}{2}}}.$$

COR. Hence the periodic time depends only on the major axis, and is therefore the same in the ellipse, and in a circle of which the diameter is equal to the major axis of the ellipse.

If PQ (see fig., p. 93) be an arc described in any time T, PQ is *ultimately* $= T \cdot V$, V being the velocity at P. Also the area SPQ is *ultimately* $= A \cdot T$. But $2SPQ$ is *ultimately* $= SY \cdot PQ$. Therefore $SY \cdot V \cdot T = 2A \cdot T$, and hence $V = \frac{2A}{SY}$.

PROP. XVI.

On the same suppositions, the velocities of the bodies at any points, are inversely as the perpendiculars from the centre of force upon the tangents at these points, and directly as the square roots of the Latera recta of the orbits.

By Prop. 14, $A = \dfrac{\sqrt{\mu} \cdot \sqrt{L}}{2\sqrt{2}}$;

hence, by Cor. to Prop. 15, $V = \dfrac{\sqrt{\mu}}{\sqrt{2}} \cdot \dfrac{\sqrt{L}}{SY}$,

and the velocity is as $\dfrac{\sqrt{L}}{SY}$.

COR. 1. Hence $L = \dfrac{2}{\mu} \cdot V^2 \cdot SY^2$, and $L \propto V^2 \cdot SY^2$.

COR. 2. At the greatest and least distances, the curve is perpendicular to the ray; SY coincides with the distance, (as SA,) and $V \propto \dfrac{\sqrt{L}}{SA}$.

COR. 3. To compare the velocity at any point of a conic section, with the velocity of a body revolving in a circle at the same distance, and acted on by the same force.

The velocity in any curve is equal to that acquired by falling down $\frac{1}{4}$ the chord of curvature, by the force, continued constant. (Prop. 6, Cor. 3.) Hence, PV being the chord of curvature,

$$V^2 \text{ in conic section} : V^2 \text{ in circle} :: 2F \cdot \frac{PV}{4} : 2F \cdot \frac{SP}{2}$$

$$:: PV : 2SP.$$

Now in the ellipse or hyperbola, $PV = \dfrac{2CD^2}{AC} = \dfrac{2SP.HP}{AC}$.

$$\text{Hence } V^2 \text{ in conic section} : V^2 \text{ in curve} :: \frac{2.SP.HP}{AC} : 2SP$$

$$:: HP : AC.$$

COR. 4. At the mean distance in the ellipse, $HP = AC$; hence the velocity is equal to that in the circle.

COR. 5. In the same figure, or in different figures, for which the latus rectum is the same, $V \propto \frac{1}{SY}$.

COR. 6. In the parabola, $SY = \sqrt{SP.SA}$, hence

$$V \propto \frac{1}{SY} \propto \frac{1}{\sqrt{SP.SA}} \propto \frac{1}{\sqrt{SP}} \text{ because } SA \text{ is constant.}$$

In the ellipse,

$$SY^2 = BC^2 \frac{SP}{HP} \text{ (See B. v.)} = BC^2 \frac{SP}{2AC - SP},$$

$$V \propto \frac{1}{SY} \propto \frac{\sqrt{2AC - SP}}{BC\sqrt{SP}}.$$

Hence, when SP diminishes, V increases in proportion to $\frac{1}{\sqrt{SP}}$, in consequence of the denominator: but it also increases in consequence of the numerator, for $2AC - SP$ increases; hence the velocity increases faster than in proportion to $\frac{1}{\sqrt{SP}}$.

In the hyperbola,

$$SY^2 = BC^2 \frac{SP}{HP} = BC^2 \frac{SP}{2AC + SP} \text{ (B. v.)},$$

$$V \propto \frac{1}{SY} \propto \frac{\sqrt{2AC + SP}}{BC\sqrt{SP}}.$$

Hence, when SP diminishes, V increases in proportion to $\frac{1}{\sqrt{SP}}$ in consequence of the denominator: but it diminishes in consequence of the numerator, for $2AC + SP$ diminishes: hence the velocity increases slower than in proportion to $\frac{1}{\sqrt{SP}}$.

Cor. 7. In the parabola, $PV = 4SP$. Hence, by Cor. 3,

$$V^2 \text{ in parabola} : V^2 \text{ in circle} :: 2 : 1,$$

$$\text{and } V \text{ in parabola} : V \text{ in circle} :: \sqrt{2} : 1.$$

In the ellipse, $HP = 2AC - SP$; hence, by Cor. 3,

$$V \text{ in ellipse} : V \text{ in circle} :: \sqrt{2AC - SP} : \sqrt{AC}$$

$$:: \sqrt{2 - \frac{SP}{AC}} : 1.$$

Hence the ratio is less than $\sqrt{2} : 1$.

In the hyperbola, $HP = 2AC + SP$; hence, by Cor. 3,

$$V \text{ in the hyperbola} : V \text{ in circle} :: \sqrt{2AC + SP} : \sqrt{AC}$$

$$:: \sqrt{2 + \frac{SP}{AC}} : 1.$$

Hence the ratio is greater than $\sqrt{2} : 1$.

Cor. 8. In the same way as in Cor. 3,

$$V \text{ in conic section} : V \text{ in circle of radius } \tfrac{1}{2}L :: \frac{\sqrt{L}}{SY} : \frac{\sqrt{L}}{\frac{1}{2}L}$$

$$:: \tfrac{1}{2}L : SY,$$

by Cor. 5, for the latus rectum L is the same for both curves.

Cor. 9. By the same reasoning,

V in circle of radius $\frac{1}{2}L$: V in circle of radius SP :: $\sqrt{SP} : \sqrt{\frac{1}{2}L}$. Prop. 6. Cor. 3. Compounding this with the Proposition in last Cor.,

$$V \text{ in conic section at } P : V \text{ in circle of radius } SP$$

$$:: \sqrt{SP \cdot \tfrac{1}{2}L} : SY.$$

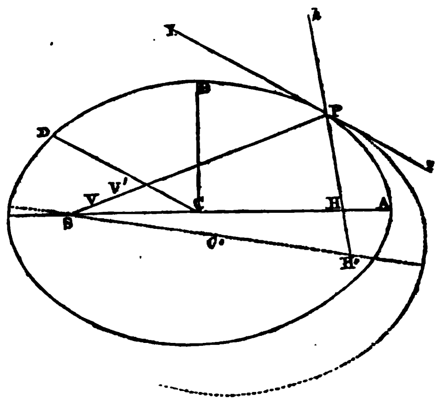
PROP. XVII.

A given force being inversely as the square of the distance from the centre, and the velocity and direction of the motion at a given point being known; to determine the curve described.

Let the force at P be $\frac{\mu}{SP^2}$, and V being the velocity at P, assume PV so that

$$V^2 = \frac{2\mu}{SP^2} \cdot \frac{PV}{4}$$

or
$$PV = \frac{2V^2 SP^2}{\mu}.$$



Draw PH making the angle $ZPH = YPS$.

If PV be less than $4SP$, take PH on the same side of YZ as S, and such that

$$HP = \frac{SP \,.\, PV}{4SP - PV}, \text{ whence } PV = \frac{4SP \,.\, HP}{SP + HP}.$$

If, then, we construct an ellipse with foci S and H, PV will be its chord of curvature through S; for, CD being conjugate to CP, chord of curvature

$$= \frac{2CD^2}{AC} = \frac{4CD^2}{2AC} = \frac{4SP \,.\, HP}{SP + HP} \text{ by Conics.}$$

If $PV = 4SP$, produce HP on the other side of YZ, take $PX = PS$, and draw a line WX perpendicular to PX. If we construct a parabola with focus S and directrix WP, the chord of curvature at P will be $4SP$ or PV.

If PV be greater than $4SP$, take Ph in HP produced, such that $hP=\frac{SP.PV}{PV-4SP}$, whence $PV=\frac{4SP.hP}{hP-SP}$. If therefore we construct a hyperbola with foci S and h, PV will be its chord of curvature through S. For, CD being conjugate to CP, chord of curvature $=\frac{4CD^2}{2AC}=\frac{4SP.hP}{hP-SP}$ by Conics.

And the conic section found by this construction will be the curve which the body describes: for if the body move in this conic section and have the velocity V at P, the force at P will be $\frac{\mu}{SP^2}$, by Prop. 6, Cor. 4. And there cannot be two curves which a body may describe, departing from P with the same velocity in the same direction, and acted on by a force of the same magnitude, and varying according to the same law: for if this were supposed, there would be nothing to determine which of the two curves the body should describe in any given instance. Therefore the conic section just found will be the path of the body.

Cor. 1. If the direction of motion at P be perpendicular to SP, P is the vertex of the curve, $PV=\frac{2CD^2}{AC}=\frac{2BC^2}{AC}=L$. And $HP=\frac{SP.L}{4SP-L}$ in the ellipse; $hP=\frac{SP.L}{L-4SP}$ in the hyperbola.

Cor. 2. If the velocity at the vertex be V, since at that point $SY=SP$, by Prop. 16, $L=\frac{2}{\mu}V^2.SP^2$; and the orbit may be constructed by the last corollary.

Cor. 3. If the velocity of the body at any point be altered, the change in the orbit may be found by the proposition.

Thus at the point P of an ellipse, Fig., p. 101, let the velocity be altered in the ratio $1:n$, the force remaining the same. Since the direction of the motion is supposed to be unchanged, PH will be in the same direction as before. Let F be the force, V the velocity at P, then, by the change, V becomes nV; let PV become PV', and PH become PH'. Now

$$V^2 = 2F.\frac{PV}{4},\quad n^2V^2 = 2F\frac{PV'}{4},\quad \text{whence } n^2.PV = PV'.$$

But $PV = \dfrac{4SP.HP}{SP+HP}$, therefore $PV' = \dfrac{4n^2SP.HP}{SP+HP}$.

And $H'P = \dfrac{SP.PV'}{4SP - PV'} = \dfrac{4n^2.SP^2.HP}{4SP(SP+HP) - 4n^2SP.HP}$,

$$\frac{H'P}{HP} = \frac{n^2SP}{SP - (n^2-1)HP}.$$

And the position of H' being found, SH' the new position of the major axis is known.

Also, we have $H'P = \dfrac{n^2SP.HP}{SP - (n^2-1)HP}$,

$$SP + H'P = \frac{SP^2 + SP.HP}{n^2SP - (n^2-1)(SP+HP)}.$$

(Since $SP + HP = 2AC$),

$$\frac{SP + H'P}{2AC} = \frac{SP}{n^2SP - (n^2-1)2AC}.$$

And $SP + H'P$ is the new axis major.

The new centre C' is the bisection of SH', and the new ellipse is determined.

Cor. 4. If the force which acts on the body at any point be altered, the change in the orbit may be found by the proposition.

At the point P of an ellipse, Fig., p. 101, let the force be altered in the ratio 1 : m, the velocity remaining the same. As in last Cor. the direction of the motion being supposed to be unchanged, PH will continue the same direction as at first. Let F the force become mF, and PV, PH become PV', PH'. Then

$$V^2 = 2F.\frac{PV}{4},\quad V^2 = 2mF.\frac{PV'}{4},\ \text{whence } PV = m.PV'.$$

$$\text{But } PV = \frac{4SP.HP}{SP+HP},\ \text{therefore } PV' = \frac{4SP.HP}{m(SP+HP)}.$$

$$\text{And } H'P = \frac{SP.PV'}{4SP-PV'} = \frac{4SP^2.HP}{4mSP(SP+HP)-4SP.HP},$$

$$\frac{H'P}{HP} = \frac{SP}{mSP-(1-m)HP}.$$

Hence the new position SH' of the major axis is found.

$$\text{Also, we have } H'P = \frac{SP.PH}{mSP-(1-m)HP},$$

$$SP + H'P = \frac{mSP^2 + mSP.HP}{SP-(1-m)(SP+HP)}.$$

And since $SP + HP = 2AC$,

$$\frac{SP+H'P}{2AC} = \frac{mSP}{SP-(1-m)2AC}.$$

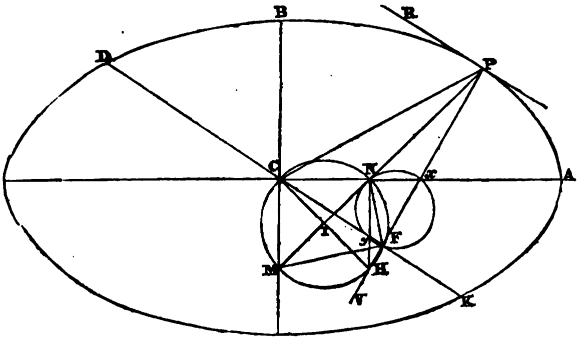
And $SP + H'P$ is the new axis major.

The new centre C' is the bisection of SH' and the new ellipse is determined.

PROP. XVIII.

Having given, at a given point, the velocity and direction of the motion of a body, revolving about a given centre of force varying as the distance, to find the curve described.

Let *PR* be the given direction of projection, *C* being the centre. And let *PV* be four times the space through which a body must fall, to acquire, by the action of the force at *P*, ($= \mu . CP$) continued constant, the velocity which the body has at *P*. Hence *PV* is given by the formula for constant forces, $V^2 = 2F . \frac{PV}{4} = \frac{\mu . CP . PV}{2}$.



Draw *DCK* parallel to *PR*, *PF* perpendicular on it, and take in *PF*, *PH* so that $PH^2 = \frac{CP . PV}{2}$. Join *CH*; bisect it in *I*; join *PI*; take *IM*, *IN* each equal to *IC* or *IH*. Draw *CN*, *CM*; and in these lines take $CA = PM$, $CB = PN$.

CA, *CB* are the positions and magnitudes of the semi-axes major and minor of the ellipse described.

By Conics, if *PV* be the chord of curvature at *P*, and *CD* the conjugate semi-diameter,

$$PV = \frac{2CD^2}{CP}, \text{ whence } CD^2 = \frac{CP . PV}{2} = PH^2.$$

With centre *I* and radius *IC* describe a circle; it will pass through *C*, *H*, *M*, *N*, and *F* because *CFH* is a right angle. Therefore $PF . PH = PM . PN = CA . CB$.

$$\begin{aligned} \text{Also } PC^2 + PH^2 &= PI^2 + IC^2 + PI^2 + IH^2, \\ &= PI^2 + IM^2 + PI^2 + IN^2, \\ &= (PI + IM)^2 + (PI - IN)^2, \\ &= PM^2 + PN^2, \\ &= CA^2 + CB^2. \end{aligned}$$

Hence, since $CD = PH$, we have

$$PF \,.\, CD = AC \,.\, CB, \quad PC^2 + CD^2 = AC^2 + CB^2.$$

And these are two equations which determine AC, BC. But these are two properties of the semi-axes major and minor. (Conics.) Therefore AC and BC have the due magnitude of the semi-axes of the ellipse, of which the chord of curvature at P is PV.

Also they have the due position. For let PH, CN meet in x, and CF, NH in y. The angles xFy, xNy being right angles, the circle on diameter xy will pass through F and N.

Also

angle $PNx = CNM = CFM = xFN$, because $CFx = MFN$.

Therefore PNx is equal to the angle in the alternate segment xFN, and therefore PN is a tangent to the circle xy at N.

Hence $PF.Px = PN^2 = BC^2$. But (B. III. Prop. 11) this is true if Cx be the position of the major axis. Therefore CA is the position of the major axis: and NCM is a right angle; therefore CB is the position of the minor axis.

Therefore the ellipse, of which PV is the chord of curvature, is rightly determined by the preceding construction. But the curve described by the body can be no other than this ellipse; for a body acted on by the force and having at P the velocity and direction which are here supposed, may move in an ellipse, of which the chord of curvature will be PV (Prop. 6. Cor. 5), and there cannot be two curves described by two bodies setting out with the same velocity, in the same direction from the same point, and acted on by the same force.

PROP. XIX.

To compare the velocity of a body moving in any curve, with the velocity of a body moving at the same point in a circle having the centre of force for its centre.

Let the chord of curvature of the curve through the centre of force be PV.

And since the chord of curvature of the circle is $2SP$, if V be the velocity in the curve at P, and U in the circle at P; by Prop. 6. Cor. 5, $V^2 = 2F.\frac{1}{4}PV$, $U^2 = 2F.\frac{1}{2}SP$.

$$\text{Hence } V^2 : U^2 :: \tfrac{1}{4}PV : \tfrac{1}{2}SP :: PV : 2SP,$$

$$\text{and } V : U :: \sqrt{PV} : \sqrt{2SP}.$$

COR. 1. In the ellipse about the centre, $PV = \frac{2CD^2}{CP}$ (B. III. Prop. 20).

Hence, putting CP for SP,

$$V : U :: CD : CP.$$

COR. 2. In the ellipse about the focus, $PV = \frac{2CD^2}{AC}$ (B. III. Prop. 20.)

$$= \frac{2SP.HP}{AC}.$$

$$\text{Hence } V : U :: \sqrt{\frac{2SP.HP}{AC}} : \sqrt{2SP} :: \sqrt{HP} : \sqrt{AC}.$$

Cor. 3. In the equiangular spiral, $PV = 2SP$ (B. v.).

$$\text{Hence } V : U :: \sqrt{2SP} : \sqrt{2SP}:$$

the velocity in the spiral is equal to that in the circle.

Prop. XX.

To find the angular velocity of a body moving in a curve acted upon by any force.

The angular velocity is measured by the ultimate quotient of the angle described, in any small time, divided by the time.

Hence in the diagram, p. 82,

$$\text{angular velocity} = \frac{PSQ}{T} = \frac{A\,.\,\text{angle } PSQ}{A\,.\,T} = \frac{A\,.\,\text{angle } PSQ}{\text{area } PSQ}$$

$$= \frac{A\,.\,\dfrac{QT}{SQ}}{\frac{1}{2}SP\,.\,QT} \text{ ultimately} = \frac{2A}{SP^2}.$$

Hence the angular velocity is inversely as SP^2.

BOOK V.

THE DIFFERENTIAL CALCULUS.

DEFFINITIONS.

1. *The Differential Calculus* is a system of symbolical calculation, the object of which is to determine the ultimate ratios of the differences of quantities, and to solve various problems by means of these ultimate ratios.

2. The quantities are here considered as *variable*, and depending upon each other by means of algebraical and transcendental operations of calculation. A quantity y which thus depends upon a quantity x, is called a *function* of x; and x is called *the independent variable.*

3. *Explicit functions* are those in which the operations to be performed upon the independent variable are denoted by the symbols, according to the order in which they must be performed to give the function. *Implicit functions* are those in which the function and the independent variable are involved in an equation or equations, which is not solved so as to give the function by a direct series of operations.

In the following cases y is an explicit function of x:

$$y = mx^2, \quad y = \sqrt{(a + bx + cx^2)}, \quad y = \frac{ax^m + bx^n}{\sqrt{(1 + 2px + x^2)}},$$

$$y = a^x, \quad y = \sin x, \quad y = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad y = \sqrt{1 - x^2}\, e^{\sin x}.$$

In the following cases y is an implicit function of x; $y^2 - 2mxy + x^2 = 0$; $y^3 - 3xy + x^3 = a^3$; $y^m - axy = b$; $e^y - yx = c$; $y - x \sin y = a$.

In these cases the quantities a, b, c, e, m, p are considered as *constant*, while x and y are variable. In such cases as that in which $y = mx^2$, m is called a *parameter*.

If y be a function of x, explicit or implicit, x is a function of y.

4. If when y is a function of x, x become x', and y in consequence become y', $x' - x$ and $y' - y$ are the *increments* or *differences* of x and y.

5. The *Differential Coefficient* of y *with reference to* x, is the ultimate value of the fraction $\frac{y' - y}{x' - x}$; and is denoted by the symbol $\frac{dy}{dx}$.

6. From this definition are to be deduced the Rules of Differentiation of all functions, algebraical and transcendental; and the application of Differential Coefficients to problems respecting curves, expansions, &c.

RULES OF DIFFERENTIATION OF ALGEBRAICAL FUNCTIONS.

7. Differential Coefficients are in general finite quantities, being the limits of the ratios of two vanishing quantities.

8. Let p, q, y be any functions of x, and when x becomes $x + h$, let p become $p + \alpha$, q become $q + \beta$, and y become $y + k$: then,

$$\frac{dp}{dx} = \text{the limit of } \frac{\alpha}{p}, \quad \frac{dq}{dx} = \text{the limit of } \frac{\beta}{h},$$

$$\frac{dy}{dx} = \text{the limit of } \frac{k}{h}.$$

This appears from Art. 5.

9. If p, q be any functions of x, and $y = pq$; we shall have

$$\frac{dy}{dx} = q\frac{dp}{dx} + p\frac{dq}{dx}.$$

For let x, p, q, y, become $x+h$, $p+\alpha$, $q+\beta$, $y+k$: then

$$y + k = (p + \alpha)(q + \beta) = pq + q\alpha + p\beta + \alpha\beta, \quad y = pq;$$

$$\text{hence, } k = q\alpha + p\beta + \alpha\beta,$$

$$\frac{k}{h} = q\frac{\alpha}{h} + p\frac{\beta}{h} + \frac{\alpha}{h}\beta:$$

and when h vanishes, $\frac{\alpha}{h}$ remains finite (Art. 7), but β vanishes; hence, $\frac{\alpha}{h}\beta$ vanishes. Also, $\frac{k}{h}$, $\frac{\alpha}{h}$, $\frac{\beta}{h}$ become

$$\frac{dy}{dx}, \frac{dp}{dx}, \frac{dq}{dx}. \quad \text{(Art. 8.)}$$

$$\text{Hence, } \frac{dy}{dx} = q\frac{dp}{dx} + p\frac{dq}{dx}.$$

10. Multiply this equation by the other, $\frac{1}{y} = \frac{1}{pq}$, and we have

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{p}\cdot\frac{dp}{dx} + \frac{1}{q}\cdot\frac{dq}{dx}.$$

11. If p, q, r, be any functions of x, and $y = pqr$, we shall have

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{p}\cdot\frac{dp}{dx} + \frac{1}{q}\cdot\frac{dq}{dx} + \frac{1}{r}\cdot\frac{dr}{dx}.$$

For let $pq = s$; then $y = pqr = sr$; and hence, by Art 10, $\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{s}\cdot\frac{ds}{dx} + \frac{1}{r}\cdot\frac{dr}{dx}$; but by the same article, since $s = pq$,

$$\frac{1}{s}\cdot\frac{ds}{dx} = \frac{1}{p}\cdot\frac{dp}{dx} + \frac{1}{q}\cdot\frac{dq}{dx};$$

hence, $\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{p}\cdot\frac{dp}{dx} + \frac{1}{q}\cdot\frac{dq}{dx} + \frac{1}{r}\cdot\frac{dr}{dx}.$

12. In the same manner, if $y = pqrs\ldots(n \text{ factors})$, we shall have

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{p}\cdot\frac{dp}{dx} + \frac{1}{q}\cdot\frac{dq}{dx} + \frac{1}{r}\cdot\frac{dr}{dx} + \frac{1}{s}\cdot\frac{ds}{dx} + \ldots \text{ (}n\text{ terms).}$$

13. If $y = p^m$, $\frac{dy}{dx} = mp^{n-1}\frac{dp}{dx}$.

For in the last article, let q, r, $s\ldots$ each $= p$: therefore,

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{p}\cdot\frac{dp}{dx} + \frac{1}{p}\cdot\frac{dp}{dx} + \text{\&c. (}n\text{ terms)}$$

$$= \frac{n}{p}\cdot\frac{dp}{dx}$$

$$\frac{dy}{dx} = \frac{ny}{p}\cdot\frac{dp}{dx} = np^{n-1}\frac{dp}{dx}, \text{ because } y = p^n.$$

14. If $y = x^n$ (n being a whole number), $\frac{dy}{dx} = nx^{n-1}$.

For if $p = x$, $p + a = x + h$, $a = h$.

Hence $\frac{a}{h} = 1$, whence $\frac{dp}{dx} = 1$, by Art. 8.

Whence by last article,

$$\frac{dy}{dx} = nx^{n-1}.$$

15. If $y = x^{\frac{m}{n}}$, $\frac{m}{n}$ being positive or negative,

$$\frac{dy}{dx} = \frac{m}{n}x^{\frac{m}{n}-1}.$$

For let $u = x^m$, therefore $u = x^m = y^n$.

$$\text{And } \frac{du}{dx} = mx^{m-1}, \text{ by Art. 14.}$$

$$\text{Also } \frac{du}{dx} = ny^{n-1}\frac{dy}{dx}, \text{ by Art. 13.}$$

Hence $ny^{n-1}\frac{dy}{dx} = mx^{m-1}$; or $n\frac{y^n}{y}.\frac{dy}{dx} = m\frac{x^m}{x}$; but $y^n = x^m$;

$$\text{therefore, } \frac{dy}{dx} = \frac{m}{n}.\frac{y}{x} = \frac{m}{n}x^{\frac{m}{n}-1}.$$

16. If $y = p^{\frac{m}{n}}$, p being a function of x,

$$\frac{dy}{dx} = \frac{m}{n}p^{\frac{m}{n}-1}\frac{dp}{dx}.$$

For let $y^n = p^m = u$.

$$\text{Then } \frac{du}{dx} = ny^{n-1}\frac{dy}{dx} = mp^{m-1}\frac{dp}{dx}, \text{ by Art. 14;}$$

$$\text{or } n\frac{y^n}{y} \cdot \frac{dy}{dx} = m\frac{p^m}{p} \cdot \frac{dp}{dx}\text{: but } y^n = p^m.$$

$$\text{Therefore } \frac{dy}{dx} = \frac{m}{n} \cdot \frac{y}{p} \cdot \frac{dp}{dx} = \frac{m}{n}p^{\frac{m}{n}-1}\frac{dp}{dx}.$$

17. If $y = p + c$, p being any function of x, and c positive or negative,

$$\frac{dy}{dx} = \frac{dp}{dx}.$$

For when x becomes $x + h$, if p becomes $p + a$, and y becomes $y + h$, $y + k = p + a + c$, $k = a$, $\frac{k}{h} = \frac{a}{h}$, and at the limit,

$$\frac{dy}{dx} = \frac{dp}{dx}.$$

18. If $y = cp$, p being any function of x,

$$\frac{dy}{dx} = c\,\frac{dp}{dx}.$$

For making the same suppositions

$$y + k = c\,(p + a) = cp + ca;$$

whence $k = ca$; $\frac{n}{h} = c\,\frac{a}{h}$; and taking the limit $\frac{dy}{dx} = c\,\frac{dp}{dx}$:

$$\text{Hence if } y = -p,\ \frac{dy}{dx} = -\frac{dp}{dx}.$$

19. If y decrease when x increases,

$$\frac{dy}{dx} = -\text{ limit of } \frac{\text{decrement of } y}{\text{increment of } x}.$$

For let x, y become $x + h$, $y - k'$; and let $-k' = k$.

Hence x, y become $x + h$, $y + k$; and by Art. 5.

$$\frac{dy}{dx} = \text{limit of } \frac{k}{h} = \text{limit of } -\frac{k'}{h} = -\text{ limit of } \frac{k'}{h}.$$

20. If y be any function of p and p any function of x,

$$\frac{dy}{dx} = \frac{dy}{dp}\cdot\frac{dp}{dx}.$$

For if h, a, k be the increments of x, p, y,

$$\frac{dy}{dx} = \text{limit of } \frac{k}{h} = \text{limit of } \frac{k}{a}\cdot\frac{a}{h} = \text{limit of } \frac{k}{a} \text{ limit of } \frac{a}{h}.$$

But the limit of $\frac{k}{a}$ is $\frac{dy}{dp}$, when p is supposed to be the independent variable; and the limit of $\frac{a}{h}$ is $\frac{dp}{dx}$; hence

$$\frac{dy}{dx} = \frac{dy}{dp}\cdot\frac{dp}{dx}.$$

21. EXAMPLES of Arts. 15, 16, 17, 18.

(1) Let $y = (a + bx^m)^n$,

$$\frac{dy}{dx} = n(a + bx^m)^{n-1} \frac{d.(a + bx^m)}{dx}, \text{ by Art. 16},$$

$$= n(a + bx^m)^{n-1} \frac{d.bx^m}{dx}, \text{ by Art. 17},$$

$$= n(a + bx^m)^{n-1} b.\frac{d.x^m}{dx}, \text{ by Art. 18},$$

$$= mnbx^{m-1}(a + bx^m)^{n-1}, \text{ by Art. 14.}$$

(2) Let $y = \sqrt{(a^{\frac{1}{2}}x^{\frac{1}{2}} + b^{\frac{2}{3}}x^{\frac{1}{3}})}$,

$$\frac{dy}{dx} = \tfrac{1}{2}(a^{\frac{1}{2}}x^{\frac{1}{2}} + b^{\frac{2}{3}}x^{\frac{1}{3}})^{-\frac{1}{2}} \times \frac{d(a^{\frac{1}{2}}x^{\frac{1}{2}} + b^{\frac{2}{3}}x^{\frac{1}{3}})}{dx}, \text{ by Art. 16},$$

$$= \tfrac{1}{2}(a^{\frac{1}{2}}x^{\frac{1}{2}} + b^{\frac{2}{3}}x^{\frac{1}{3}})^{-\frac{1}{2}} \times (\tfrac{1}{2}a^{\frac{1}{2}}x^{-\frac{1}{2}} + \tfrac{1}{3}b^{\frac{2}{3}}x^{-\frac{2}{3}})$$

$$= (a^{\frac{1}{2}}x^{\frac{1}{2}} + b^{\frac{2}{3}}x^{\frac{1}{3}})^{-\frac{1}{2}} \times \frac{3a^{\frac{1}{2}}x^{\frac{2}{3}} + 2b^{\frac{2}{3}}x^{\frac{1}{2}}}{12x^{\frac{7}{6}}}$$

$$= \frac{3a^{\frac{1}{2}}x^{\frac{1}{2}} + 2b^{\frac{2}{3}}x^{\frac{1}{3}}}{12x\sqrt{(a^{\frac{1}{2}}x^{\frac{1}{2}} + b^{\frac{2}{3}}x^{\frac{1}{3}})}}.$$

22. If y be a function of x, the differential coefficient $\frac{dy}{dx}$ is a function of x, and may have its differential coefficient taken with respect to x; this differential coefficient is called the *second differential coefficient*, and is denoted thus, $\frac{d^2y}{dx^2}$.

In like manner the differential coefficient of the second differential coefficient is called the *third differential coefficient;* and is denoted thus, $\frac{d^3y}{dx^3}$; and so on.

23. These differential coefficients are by some modern writers denoted thus; d_xy, d_x^2y, d_x^3y, &c.

24. Examples of Art. 22:

(1) Let $y = x^n$;

$$\frac{dy}{dx} = nx^{n-1};$$

$$\frac{d^2y}{dx^2} = n.\overline{n-1}\,x^{n-2};$$

$$\frac{d^3y}{dx^3} = n.\overline{n-1}.\overline{n-2}\,x^{n-3};$$

and so on.

(2) Let $y = x^m (a-x)^n$;

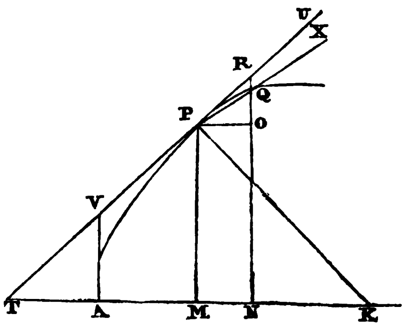
$$\frac{dy}{dx} = mx^{m-1}(a-x)^n - nx^m (a-x)^{n-1};$$

$$\frac{d^2y}{dx^2} = m.\overline{m-1}\,x^{m-2}(a-x)^n - 2mnx^{m-1}(a-x)^{n-1} + n.\overline{n-1}\,x^m (a-x)^{n-2};$$

and so on.

APPLICATION OF FIRST DIFFERENTIAL COEFFICIENTS TO CURVES DEFINED BY RECTANGULAR CO-ORDINATES.

25. If a curve be given by an equation between its co-ordinates x, y, the differential coefficient $\frac{dy}{dx}$ is the (trigonometrical) tangent of the angle which the curve makes with the line of abscissas x at any point.



Let P be any point in the curve, Q a point in the curve near P; PM, QN ordinates, PO parallel to AM; PU the tangent meeting NQ in R.

Let x, y become

$x + h$, $y + k$:

therefore $PO = h$, $OQ = k$:

and $\frac{k}{h} = \frac{QO}{OP}$.

Therefore Limit of $\frac{k}{h}$ = Limit of $\frac{QO}{OP}$.

But Limit of $\frac{k}{h} = \frac{dy}{dx}$, by Art. 5 of this Book.

And Limit of $\frac{QO}{OP} = \frac{OR}{OP}$, by Prop. 8. B. II.

Therefore $\frac{dy}{dx} = \frac{OR}{OP} = \tan UPO$.

26. We may hence draw a tangent to a given curve.

27. If the tangent UP meet the line of abscissas in T, MT is the *subtangent*.

$TM . \tan PTM = PM$; hence the subtangent $= \dfrac{y}{\frac{dy}{dx}}$;

28. A *normal* is a line perpendicular to the curve, as PK.

The (trigonometrical) tangent of the angle which the normal makes with the line of abscissas $= \dfrac{1}{\frac{dy}{dx}}$.

Because it is perpendicular to the tangent, by B. I. Art. 32.

29. If the normal meet the abscissa in K, MK is the *subnormal*.

The subnormal $= y\frac{dy}{dx}$.

30. To find at what angle the curve cuts the axis.

In the differential coefficient $\frac{dy}{dx}$, put for x the value which makes $y = 0$.

When $\frac{dy}{dx} = 0$, the curve is parallel to the line of abscissas x; and when $\frac{dy}{dx}$ is infinite, the curve is parallel to the line of ordinates y.

31. To find whether a given curve have asymptotes.

$$\text{By Art. 27, } AT = \frac{y}{\frac{dy}{dx}} - x.$$

If this be finite when x is infinite, the tangent AT has a limiting position in which AT is finite, and the curve has an asymptote.

Let AV, parallel to the ordinates, meet the tangent in V,

$$AV = AT \cdot \tan PTA = y - x\frac{dy}{dx}.$$

If this be finite when x is infinite, the curve has an asymptote, for the same reason as before.

Hence, in order to determine whether the curve have an asymptote, we must find whether either of the above quantities is finite when x is infinite.

32. Find the subtangents, and subnormals, of the curves defined by the following equations, and the angles at which they cut the axis:

The cubical parabola ..... $y^3 = a^2x$.

The semi-cubical parabola $y^3 = ax^2$.

The general parabola..... $y^{m+n} = a^m x^n$.

The knot.................. $ay = x\sqrt{a^2 - x^2}$.

The witch.................. $xy = a\sqrt{ax - x^2}$.

The lemniscata............ $(x^2+y^2)^2 = a^2(x^2 - y^2)$.

The cissoid .... $y^2(a - x) = x^3$.

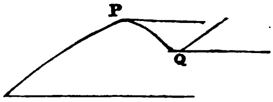
The conchoid.............. $x^2y^2 = (a + x)^2(b^2 - x^2)$.

33. A quantity (y) which depends upon another (x) is a *maximum* for a certain value (X) of x, when the value of y corresponding to x is greater than all other values within a certain finite interval including x; (for instance, greater than all values of y corresponding to values of x from $X - a$ to $X + a$, a being finite.)

A quantity, in like manner, is a *minimum*, when it is less than the values immediately adjacent on each side.

34. Axiom. If a quantity first increase and then decrease, it comes to a maximum at the point where the increase ends and the decrease begins. If a quantity first decrease and then increase, it comes to a minimum at the point where the decrease ends and the increase begins.

In a curve of continued curvature, at points where the ordinate is a maximum or minimum, the tangent to the curve is parallel to the axis, as at P, Q.



35. When y, a function of x, is a maximum or minimum; $\frac{dy}{dx} = 0$.

For any function of x may be represented by the ordinate of a curve, and the function will have a maximum (or a minimum), when the ordinate has a maximum (or minimum). But the ordinate has a maximum (or minimum), when the tangent is parallel to the line of abscissas; that is, when $\frac{dy}{dx} = 0$. (Art. 28.) Therefore, &c. Q. E. D.

It will be seen hereafter, that it is not true conversely, that when $\frac{dy}{dx} = 0$, y has a maximum or minimum.

36. Examples to Art. 35. To find the maximum and minimum values of the following quantities:

The ordinate of the *knot*, in which $y = \frac{x}{a}\sqrt{a^2 - x^2}$.

The ordinate of the curve, given B. I. Art. 6, in which

$$30y = 4x^3 - 12x^2 - 63x + 30.$$

The ordinate of the *lemniscata*, in which

$$(x^2 + y^2)^2 = a^2(x^2 - y^2).$$

The rectangle inscribed in a given circle.

The cylinder inscribed in a given sphere.

The straight line which bisects a triangle cutting two given sides.

The cones whose convex surface is given (the maximum solid content).

37. If u be the area of a curve, $\frac{du}{dx} = y$.

By B. I, Prop. 3, Cor. 4, if the abscissa of a curve receive an increment AB (see diag. p. 24), the area receives an increment $ABba$, which is ultimately equal to Aa, AB. That is, if x receive an increment h and u an increment i, i is ultimately equal to yh.

Therefore $y = \frac{i}{h}$ ultimately $= \frac{du}{dx}$.

38. If u be the solid content of the solid generated by the revolution of any curve round the axis of x, $\frac{du}{dx} = \pi y^2$.

For if $u + h$ and i be corresponding increments of x and u, by reasoning from Cor. 6 of Prop. 3, B. I, instead of the Proposition, it will appear, as in Cor. 4, that i is ultimately equal to $\pi y^2 h$.

$$\text{Therefore } \pi y^2 = \frac{i}{h} \text{ ultimately} = \frac{du}{dx}.$$

39. If u be the length of the arc of a curve,

$$\frac{du}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}.$$

For if h, i be corresponding increments of x and u, we have (see diag. p. 116), $\frac{i}{h} = \frac{PQ}{PO}$;

$$\text{and limit of } \frac{PQ}{PO} = \frac{PR}{PO} \text{ (B. II. Prop. 8)}$$

$$= \frac{\sqrt{(PO^2 + OR^2)}}{PO} = \sqrt{\left\{1 + \left(\frac{OR}{OP}\right)^2\right\}}$$

$$= \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}, \text{ by Art. 23,}$$

$$\text{hence } \frac{du}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}.$$

40. If u be the curve surface of any solid generated by the revolution of any curve round the axis of x.

$$\frac{du}{dx} = 2\pi y \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}.$$

For by reasoning from Art. 13, B. II, instead of Art. 6, it will appear as in B. II, Prop. 7, that the surfaces generated by the revolution of the arc, its chord, and its tangent round any axis, are ultimately equal. Also the surface generated by the revolution of the tangent PQ (see diag. p. 116) is

$$2\pi . MP . PQ + \pi . PQ . OQ \text{ (see Errata).}$$

Hence the increment of the surface generated by the revolution of the curve AP is ultimately

$$= 2\pi . MP . PQ + \pi . PQ . OQ.$$

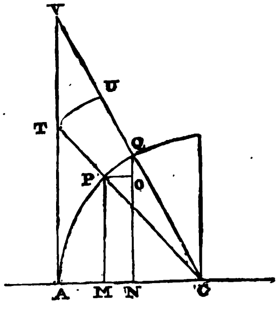
And $\dfrac{\text{increment of the surface}}{OP} = 2\pi . MP . \dfrac{PQ}{OP} + \pi \dfrac{PQ}{OP} . OQ.$

Also ultimately the first side becomes $\dfrac{du}{dx}$, the fraction $\dfrac{PQ}{OP}$ is $\sqrt{\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}}$, and the last term vanishes, because $\dfrac{PQ}{OQ}$ is finite and OQ vanishes.

Therefore $\dfrac{du}{dx} = 2\pi y \sqrt{\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}}.$

RULES OF DIFFERENTIATION OF CIRCULAR FUNCTIONS.

41. Let APQ be a circular arc of which the radius CA is 1; PM, QN the sines of the angles ACP, ACQ; AT, AV the tangents of these angles; PO parallel to AC, CU equal to CT.



Let $ACP = x$, $MP = y$, or

$y = \sin x$, then $\dfrac{\text{increment of } y}{\text{increment of } x}$

$= \dfrac{\text{increment of } MP}{\text{increment of } AP} = \dfrac{OQ}{PQ};$

and taking the limits $\dfrac{dy}{dx} = \dfrac{CM}{CP}$ (B. II. Prop. 7.) $= \cos x$.

Hence $\dfrac{d . \sin x}{dx} = \cos x.$

42. Let $y = CM = \cos x$.

$$\frac{\text{decrement of } y}{\text{increment of } x} = \frac{OP}{PQ} = \frac{MP}{CP} = \sin x.$$

$$\text{Hence } -\frac{d.\cos x}{dx} = \sin x; \text{ and } \frac{d.\cos x}{dx} = -\sin x.$$

43. Let $y = AT = \tan x$.

$$\frac{\text{increment of } y}{\text{increment of } x} = \frac{TV}{PQ} = \frac{TV}{TU} \cdot \frac{TU}{PQ}.$$

$$\text{And taking the limits, } \frac{TV}{TU} = \frac{TC}{AC}, \quad \frac{TU}{PQ} = \frac{TC}{PC} = \frac{TC}{AC}.$$

$$\text{Hence } \frac{dy}{dx} = \frac{TC^2}{AC^2} = (\sec x)^2; \text{ and } \frac{d.\tan x}{dx} = (\sec x)^2.$$

44. Let $y = CT = \sec x$.

$$\frac{\text{increment of } y}{\text{increment of } x} = \frac{UV}{PQ} = \frac{UV}{TU} \cdot \frac{TU}{PQ}.$$

$$\text{And at the limit, } \frac{UV}{TU} = \frac{AT}{AC}, \quad \frac{TU}{PQ} = \frac{CT}{AC};$$

$$\text{whence } \frac{dy}{dx} = \frac{AT.CT}{AC^2} = \tan x . \sec x,$$

$$\frac{d.\sec x}{dx} = \tan x . \sec x.$$

45. If x and y be connected by any equation, and if when x is the independent variable, and y the function, $\frac{dy}{dx} = p$, then when y is the independent variable, and x the function, $\frac{dx}{dy} = \frac{1}{p}$.

For if h, k be the corresponding increments of x, y,

$$\frac{dx}{dy} = \text{limit of } \frac{h}{k} = \text{limit of } \frac{1}{\frac{h}{k}} = \frac{1}{\text{limit of } \frac{h}{k}} = \frac{1}{\frac{dy}{dx}} = \frac{1}{p}.$$

46. When $y = \sin x$, $x = \text{arc to } \sin y$. But this may be better expressed thus, $\sin^{-1} y = x$; the symbol sin being considered as separable from the quantity y, and being transposed from one side of the equation to the other, as if it were a symbol of quantity.

47. Let $\sin^{-1} y = x$, then $y = \sin x$, $\frac{dy}{dx} = \cos x = \sqrt{1-y^2}$.

$$\frac{d \,.\, \sin^{-1} y}{dy} = \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{\cos x},$$

$$\frac{d \,.\, \sin^{-1} y}{dy} = \frac{1}{\sqrt{1-x^2}}.$$

48. Let $\cos^{-1} y = x$, then $y = \cos x$,

$$\frac{dy}{dx} = -\sin x = -\sqrt{1-y^2},$$

$$\frac{d \,.\, \cos^{-1} y}{dy} = \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = -\frac{1}{\sin x},$$

$$\frac{d \,.\, \cos^{-1} y}{dy} = -\frac{1}{\sqrt{1-y^2}}.$$

49. Let $\tan^{-1} y = x$, $y = \tan x$, $\frac{dy}{dx} = \sec^2 x = 1 + y^2$.

$$\frac{d \,.\, \tan^{-1} y}{dy} = \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{\sec^2 x},$$

$$\frac{d \,.\, \tan^{-1} y}{dy} = \frac{1}{1+y^2}.$$

50. Let $\sec^{-1} y = x$, $y = \sec x$,

$$\frac{dy}{dx} = \tan x \sec x = y\sqrt{y^2 - 1},$$

$$\frac{d \,.\, \sec^{-1} y}{dy} = \frac{dx}{dy} = \frac{1}{\dfrac{dy}{dx}} = \frac{1}{\tan x \,.\, \sec x},$$

$$\frac{d \,.\, \sec^{-1} y}{dx} = \frac{1}{y\sqrt{y^2 - 1}}.$$

51. Examples :

$$\frac{d \,.\, \sin x \cos x}{dx} = (\cos x)^2 - (\sin x)^2 = \cos 2x,$$

$$d \,.\, \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} - \frac{2 \sin^2 x}{\cos^3 x}.$$

APPLICATION OF FIRST DIFFERENTIAL COEFFICIENTS TO POLAR EQUATIONS OF CURVES.

52. If SY (p. 126) be drawn perpendicular to the tangent at P, $SP = r$ and $SY = p$.

$$\frac{1}{r} \,.\, \frac{dr}{d\theta} = \frac{\sqrt{r^2 - p^2}}{p}.$$

For $\dfrac{\text{increment of } r}{\text{increment of } \theta} = \dfrac{OQ}{\text{angle } PSQ} = SP \,.\, \dfrac{OQ}{OP}$ (see Errata).

But ultimately OPQ becomes a right-angled triangle similar to YPS: wherefore $\dfrac{OQ}{OP} = \dfrac{PY}{SY}$ ultimately.

And $PY = \sqrt{r^2 - p^2}$;

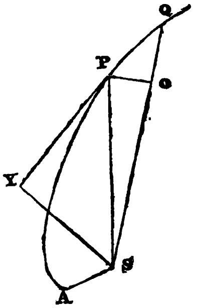
$$\therefore \quad \frac{dr}{d\theta} = r\frac{\sqrt{r^2 - p^2}}{p}.$$

53. Hence $\left(\frac{1}{r}\frac{dr}{d\theta}\right)^2 = \frac{r^2}{p^2} - 1$; $p^2 = \dfrac{r^2}{1 + \left(\frac{1}{r}\frac{dr}{d\theta}\right)^2}$:

$$\frac{r}{p} = \sqrt{\left\{1 + \left(\frac{1}{r}\frac{dr}{d\theta}\right)^2\right\}} = \sqrt{\left\{1 + \left(\frac{1}{r^2}\frac{dr}{d\theta}\right)^2\right\}}.$$

54. If $ASP = \theta$, $SP = r$, and the arc $AP = s$,

$$\frac{ds}{d\theta} = \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}.$$



For let PO be an arc described with radius SP.

$$\frac{PQ}{\text{angle } POQ} = \frac{SP \,.\, PQ}{PO};$$

and ultimately $\dfrac{PQ}{PO} = \dfrac{PS}{SY} = \dfrac{r}{p}$.

$$\text{Hence } \frac{ds}{d\theta} = \frac{r^2}{p} = r \,.\, \frac{r}{p} = \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}.$$

55. If u be the area of the spiral ASP,

$$\frac{du}{d\theta} = \tfrac{1}{2} r^2.$$

For the increment of the area ASP is ultimately equal to the sector AOP, and the sector is (Book III. Prop. 17.)

$$\tfrac{1}{2} SP \,.\, PO = \tfrac{1}{2} SP^2 \times \text{angle } PSQ,$$

because arc $PO = SP \times$ angle PSQ.

$$\text{Hence } \frac{\text{increment of } u}{\frac{1}{2} r^2 \times \text{increment of } \theta} = 1 \text{ ultimately},$$

$$\frac{\text{increment of } u}{\text{increment of } \theta} = \tfrac{1}{2} r^2 \text{ ultimately},$$

$$\text{whence } \frac{du}{d\theta} = \tfrac{1}{2} r^2.$$

56. By Art. 53, the equation between r and θ being given, the equation between r and p may be found.

Examples. To find the equation between p and r in the ellipse about the focus.

The equation to the ellipse between r and θ is

$$r = \frac{b^2}{a + ae\cos\theta} \text{ (Art. 44, B. I); whence } ae\cos\theta = 1 - \frac{ar}{b^2}:$$

$$\frac{1}{r} = \frac{a + ae\cos\theta}{b^2}; \text{ hence } -\frac{1}{r^2}\frac{dr}{d\theta} = -\frac{ae\sin\theta}{b^2}:$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{ae}{b^2} r\sin\theta; \quad \left(\frac{1}{r}\frac{dr}{d\theta}\right)^2 = \frac{a^2e^2}{b^2} r^2\sin^2\theta$$

$$= \frac{a^2e^2r^2}{b^2} - \frac{a^2e^2r^2\cos^2\theta}{b^4}$$

$$= \frac{a^2e^2r^2}{b^2} - \left(1 - \frac{ar}{b^2}\right)^2$$

$$= -1 + \frac{2ar}{b^2} - \frac{r^2}{b^2},$$

$$(\text{because } a^2e^2 - a^2 = -b^2),$$

$$\text{hence } p^2 = \frac{r^2}{1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2} = \frac{b^2r^2}{2ar - r^2} = \frac{b^2r}{2a - r}.$$

The equation to the hyperbola about the focus is

$$p^2 = \frac{b^2}{2a + r}.$$

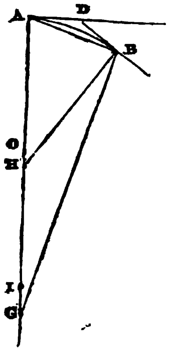
The equation to the ellipse about the centre is

$$p^2 = \frac{a^2b^2}{a^2 + b^2 - r^2}.$$

The equation to the equiangular spiral is

$$p = ar.$$

57. If, at two points in any curve, normals be drawn intersecting each other, and if these two points approach and ultimately coincide; the normal, intercepted by the intersection, is ultimately the radius of curvature.



Let A, B be two points in the curve; AD, BD tangents meeting in D; AH, BH, perpendiculars to the tangents; BG perpendicular to the chord AB, meeting AH in G.

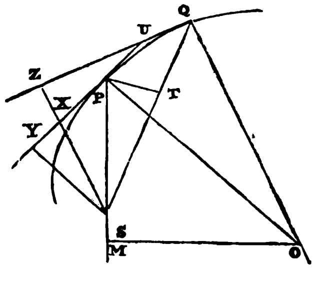
Since ABG, DBH are both right angles, the angles ABD, GBH are equal; also BAD is equal to BGA or BGH; therefore the triangles ADB, GBH are similar. Therefore

$$HB : GH :: BD : DA.$$

But ultimately BD, DA are equal (B. II. Prop. 13, Cor. 2). Hence GH, HB are ultimately equal; but HB, HA are ultimately equal; therefore GH, HA are ultimately equal; and H is ultimately the bisection of AG. But ultimately AG is equal to AI, the diameter of curvature; therefore ultimately H coincides with O, the centre of curvature, and the normal AH is ultimately the radius of curvature.

58. To find the radius of curvature of a spiral.

Let P, Q be two points, PY, QZ the tangents, PO, QO the normals; therefore PO is ultimately the radius of curvature.



Let SY, SZ be perpendicular on the tangents PY, QZ; and let SY, QZ meet in X. Then it is easily seen that ZX or $SZ - SX$ is ultimately equal to $SZ - SY$, that is, to the increment of SY. And if XP meet ZQ in U, since PO, QO are perpendicular to XP, ZQ, the angle XUZ is equal to

POQ. And the angles *UXZ*, *UZX* are ultimately right angles, as are the angles *OPQ*, *OQP*. Hence the figures *UXZ*, *OPQ* are ultimately similar.

$$\text{Wherefore } \frac{ZX}{UX} = \frac{PQ}{OP} \text{ ultimately.}$$

$$\text{But } \frac{PY}{SP} = \frac{QT}{PQ}, \text{ ultimately, } ST \text{ being equal to } SP.$$

$$\text{Also } \frac{UX}{PY} = 1 \text{ ultimately.}$$

$$\text{Hence, compounding, } \frac{ZX}{SP} = \frac{QT}{OP} \text{ ultimately.}$$

$$\text{Hence } \frac{ZX}{QT} = \frac{SP}{OP} \text{ ultimately.}$$

$$\text{But } \frac{ZX}{QT} = \frac{\text{increment of } SY}{\text{increment of } SP} = \frac{dp}{dr} \text{ ultimately.}$$

$$\text{Hence } \frac{dp}{dr} = \frac{r}{OP}; \text{ and } \frac{1}{OP} = \frac{1}{r} \cdot \frac{dp}{dr}.$$

Hence *OP* is known.

59. To find the chord of curvature passing through the pole.

Let *O* be the centre of curvature, and draw *OM* perpendicular upon *SP*. Then *PM* is half the chord of curvature. And by similar triangles *OPM*, *PSY*, $PM = \frac{SY}{SP} \cdot OP$;

$$\text{hence } \frac{1}{PM} = \frac{r}{p} \cdot \frac{1}{OP} = \frac{1}{p}\frac{dp}{dr};$$

whence $2PM$, $(PV,)$ the chord of curvature, is known.

60. EXAMPLES.

In the parabola, $p^2 = 4ar$, $\frac{2}{p}\frac{dp}{dr} = \frac{1}{r}$; hence chord of curvature $= 4r$; radius of curvature $= 2r \cdot \frac{r}{p} = \frac{r^{\frac{3}{2}}}{a^{\frac{1}{2}}}$.

In the ellipse about the centre, $p^2 = \frac{a^2 b^2}{a^2 + b^2 - r^2}$;

$$\frac{2}{p}\frac{dp}{dr} = \frac{2r}{a^2 + b^2 - r^2}; \text{ chord curv.} = \frac{2(a^2 + b^2 - r^2)}{r} = \frac{2a^2b^2}{p^2 r};$$

$$\text{radius of curvature} = \frac{r}{p}\frac{2a^2b^2}{p^2 r} = \frac{2a^2b^2}{p^3}.$$

In the equiangular spiral, $p = mr$,

$$\frac{1}{p}\frac{dp}{dr} = \frac{1}{r};$$

$$\text{chord of curvature} = 2r,$$

$$\text{radius of curvature} = \frac{r^2}{p} = \frac{r}{m}.$$

APPLICATION OF DIFFERENTIALS TO THE DIRECT PROBLEM OF CENTRAL FORCES.

61. To find the force by which a body may move in any curve defined by an equation between the ray and the perpendicular upon the tangent (the force tending to the pole).

If A be the area described in a unit of time, we have (B. IV. Prop. 6.)

$$\text{force} = \frac{8A^2}{SY^2 \, PV} = \frac{4A^2}{p^3}\frac{dp}{dr} \text{ (by Art. 59).}$$

62. To compare the velocity in the curve with the velocity in a circle at the same distance.

If V, U be the velocities, it has been shewn (B. IV. Prop. 19,) that

$$V^2 : U^2 :: PV : 2SP$$

$$:: \frac{1}{SP} : \frac{2}{PV}$$

$$:: \frac{1}{r} : \frac{1}{p}\frac{dp}{dr}.$$

63. To compare the angular velocity of SP and SY.

The angular velocity of SY is always equal to the angular velocity of PY, (diagram, p. 128,) because SY is perpendicular to PY. And the angular velocities of SP, PY are in the ultimate ratio of the angles PSQ, XUZ.

Hence,

angular velocity of SP : angular velocity of SY

$$:: \frac{PT}{SP} : \frac{XZ}{UX} \text{ ultimately}$$

$$:: \frac{UX}{SP} : \frac{XZ}{PT} \text{ ultimately}$$

$$:: \frac{PY}{SP} : \frac{XZ}{QT} \cdot \frac{QT}{PT} \text{ ultimately}$$

$$\left(\text{since } \frac{QT}{PT} = \frac{PY}{SY}\right)$$

$$:: \frac{PY}{SP} : \frac{XZ}{QT} \cdot \frac{PY}{SY}$$

$$:: \frac{SY}{SP} : \frac{\text{increment of } SY}{\text{increment of } SP}$$

$$:: \frac{p}{r} : \frac{dp}{dr}.$$

64. **Example.** Compare the angular velocity of the ray and the perpendicular on the tangent, in the ellipse about the centre.

$$p^2 = \frac{a^2b^2}{a^2+b^2-r^2}, \quad \frac{2}{p}\cdot\frac{dp}{dr} = \frac{2r}{a^2+b^2-r^2}.$$

Hence, ang. vel. of r : ang. vel. of p :: $\frac{p}{r} : \frac{pr}{a^2+b^2-r^2}$

$$:: a^2+b^2-r^2 : r^2.$$

65. In this example, the angular velocities of r and p are equal when $a^2+b^2-r^2=r^2$, or $r^2=\frac{1}{2}(a^2+b^2)$.

At this point the angular distance of p and r is a maximum.

At the same point the tangent makes the smallest angle with the radius.

66. In any curve, to find when the velocity increases fastest.

The velocity increases fastest, when the resolved part of the force in the direction of the tangent is a maximum: that is, if F be the force, when $F\frac{\sqrt{r^2-p^2}}{r}$ is a maximum.

67. **Example.** In the ellipse about the centre, to find when the velocity increases fastest.

In this case $F=\mu r$,

$$\mu r\frac{\sqrt{r^2-p^2}}{r} = \text{max.}, \quad r^2-p^2=\text{max.}, \quad 2r-\frac{2a^2b^2r}{(a^2+b^2-r^2)^2}=0,$$

$$a^2+b^2-r^2=ab, \quad r=\sqrt{(a^2-ab+b^2)}.$$

68. To find when the *paracentric velocity* is greatest.

The paracentric velocity is the velocity towards the centre.

The velocity at any point is $\frac{2A}{p}$; and resolving this in the direction of the ray, the paracentric velocity $= 2A\frac{\sqrt{r^2-p^2}}{p^2}$: And this is to be a maximum.

$\frac{2A\sqrt{r^2-p^2}}{p^2}$ is a maximum: therefore, $\frac{r^2-p^2}{p^4}$ is a max.:

$$\text{therefore, } \frac{r^2}{p^4} - \frac{1}{p^2} \text{ is a max.:}$$

$$\text{therefore, } \frac{2r}{p^4} - \frac{4r^2}{p^5} \cdot \frac{dp}{dr} + \frac{2}{p^3} \cdot \frac{dp}{dr} = 0:$$

$$\text{therefore, } rp - (2r^2 - p^2)\frac{dp}{dr} = 0.$$

69. To find when the angular velocity increases fastest.

If w be the angular velocity;

$$\text{by B. IV. Prop. 20, } w = \frac{2A}{r^2};$$

$$\text{Hence, } \frac{dw}{dt} = -\frac{4A}{r^3} \cdot \frac{dr}{dt};$$

But $-\frac{dr}{dt}$ is the paracentric velocity; and therefore,

$$= \frac{2A\sqrt{r^2-p^2}}{p^2}, \text{ by Art. 68:}$$

$$\text{Hence, } \frac{dw}{dt} = \frac{8A^2\sqrt{r^2-p^2}}{r^3p^2}.$$

This is the rate of increase; and the angular velocity increases fastest when this is a maximum; that is, when

$$\frac{r^2-p^2}{r^6p^4} \text{ is a max., or when } \frac{1}{r^4p^4} - \frac{1}{r^6p^2} \text{ is a max.}$$

70. Example. To find when the angular velocity in the parabola increases fastest. Here $p^2 = ar$. Hence,

$$\frac{1}{a^2r^6} - \frac{1}{ar^7} = \text{max.}, \quad -\frac{6}{a^2r^7} + \frac{7}{ar^8} = 0, \quad r = \frac{7a}{6}.$$

71. To find the relation of the differentials of space, velocity, and time of motion, when a body moves in a straight line to a centre of force.

In this case, if f, v, t, x be the force, velocity, time, and space described, $v = \frac{dx}{dt}$, $f = \frac{dv}{dt}$, by Prop. 10, Cor. 1, and Prop. 11, Cor. 1, of B. II. Hence, multiplying crosswise,

$$f\frac{dx}{dt} = v\frac{dv}{dt} = v\frac{dv}{dx} \cdot \frac{dx}{dt};$$

$$\text{whence } f = v\frac{dv}{dx}.$$

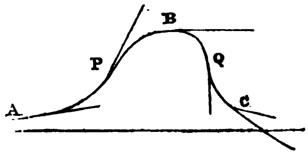
Hence f being known in terms of x, we find v by integration. And v being known, we can find t by integration from the equation $v = \frac{dx}{dt}$.

APPLICATION OF HIGHER DIFFERENTIAL COEFFICIENTS TO RECTANGULAR CURVES.

72. When in moving along a curve the ordinate constantly increases, and the angle which the tangent to the curve makes with the abscissa constantly decreases, the curve is *concave* to the line of abscissa, as from P to B.

When the ordinate increases, and *the angle of the curve* also increases, the curve is *convex* to the line of abscissas, as from A to P.

When the ordinate decreases, if the angle of the curve decrease the curve is *convex*, as from *Q* to *C*; if the angle of the tangent increase the curve is *concave*, as from *B* to *Q*.



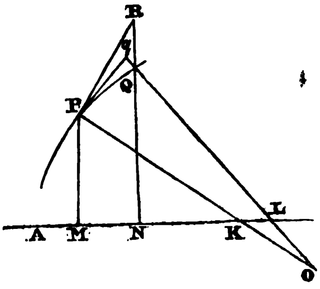
73. A point whence the curve passes from convex to concave, or the reverse, is a *point of contrary flexure*, as at *P*, *Q*.

74. At a point of contrary flexure, $\frac{d^2y}{dx^2} = 0$.

For the angle of the curve decreases to the point of contrary flexure, and increases afterwards, (or the reverse). Therefore the trigonometrical tangent of the angle of the curve decreases to that point, and afterwards increases (or the reverse); for in angles less than a right angle the tangent increases and decreases when the angle does. Therefore this tangent is at that point a minimum (or a maximum). But this tangent is $\frac{dy}{dx}$: therefore at a point of contrary flexure $\frac{dy}{dx}$ is a minimum (or a maximum). (Art. 33.) Therefore the differential coefficient of $\frac{dy}{dx}$ is 0 at this point. (Art. 35.) That is at a point of contrary flexure $\frac{d^2y}{dx^2} = 0$. (Art. 22.)

75. The radius of curvature of any curve is

$$-\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.$$



Let *AM*, *MP* be x, y, the abscissa and ordinate; *Q* a point near *P*, and *AN* its ordinate $= x'$; *PK*, *QL* two normals intersecting in *O*; the angle $PKA = \theta$,

$QLA = \theta'$. With centre O and radius OP, describe a circular arc Pq, meeting OQ;

$$\frac{Pq}{PO} = POQ = KOL = QLA - PKA.$$

$$\text{Hence } PO = \frac{Pq}{QLA - PKA} = \frac{PR}{QLA - PKA} \cdot \frac{Pq}{PR}.$$

But PR, Pq are ultimately equal, each being ultimately equal to PQ, because the tangent at P is common to both.

$$\text{Hence limit of } PO = \text{limit of } \frac{PR}{QLA - PKA};$$

$$= \text{limit of } \frac{\dfrac{PR}{MN}}{\dfrac{QLA - PKA}{MN}} = \frac{\text{limit of } \dfrac{PR}{MN}}{\text{limit of } \dfrac{QLA - PKA}{MN}}.$$

$$\text{Now limit of } \frac{PR}{MN} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}. \quad \text{(Art. 39.)}$$

$$\text{And limit of } \frac{QLA - PKA}{MN} = \text{limit of } \frac{\theta' - \theta}{x' - x} = \frac{d\theta}{dx}.$$

$$\text{But } \tan\theta = -\frac{dy}{dx}; \text{ whence } \frac{d\theta}{dx} = -\frac{\dfrac{d^2y}{dx^2}}{1 + \left(\dfrac{dy}{dx}\right)^2}. \quad \text{(Art. 49.)}$$

$$\text{Hence } PO = -\frac{\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}, \text{ the rad. curv. (Art. 57.)}$$

The radius of curvature, as given by this formula, is positive when the curve is concave to the axis, and negative when the curve is convex.

APPLICATION OF THE DIFFERENTIAL CALCULUS TO EXPANSIONS, INCLUDING DIFFERENTIATION OF EXPONENTIAL FUNCTIONS AND RESOLUTION OF TRINOMIALS INTO FACTORS.

76. A function of algebraical quantities is *expanded*, when the expressions of calculations to be performed upon compound quantities are replaced by equivalent expressions of calculations performed upon simpler quantities, forming a series of *terms* arranged according to the *indices* with which one or more of the quantities are affected.

77. If y be a function of x, and if $x + h$ be put for x, and the function expanded according to the integral powers of h, the coefficient of the simple power of h is the differential coefficient of y.

Let k be the increment which y receives when x becomes $x + h$; and let the expansion according to powers of h be $N + Ph + Qh^2 + Rh^3 +$ &c.

Therefore $y + k = N + Ph + Qh^2 + Rh^3 +$ &c.

But since the expansion is equivalent to the function, for all values, let $h = 0$, whence $k = 0$ and $y = N$.

$$\text{Hence, } k = Ph + Qh^2 + Rh^3 + \text{\&c.}$$

$$\frac{k}{h} = P + Qh + Rh^2 + \text{\&c.}$$

And when h, the increment of x, vanishes, this equation becomes $\dfrac{dy}{dx} = P$, the coefficient of h.

78. If $y = a^x$, and if $A = a - 1 - \frac{1}{2}\overline{a-1}^2 + \frac{1}{3}\overline{a-1}^3 -$ &c.

$$\frac{dy}{dx} = a^x A.$$

For let x become $x+h$, and y become $y+k$; then

$$y+k=a^{x+h}=a^x.a^h$$

$$=a^x(1+\overline{a-1})^h$$

$$=a^x\{1+h.\overline{a-1}+\frac{h.\overline{h-1}}{1.2}\overline{a-1}^2+\frac{h.\overline{h-1}\,\overline{h-2}}{1.2.3}\overline{a-1}^3+\&c.\}$$

Expand $\frac{h.\overline{h-1}}{1.2}$, $\frac{h.\overline{h-1}.\overline{h-2}}{1.2.3}$, &c. by multiplication; and it is evident that the last terms of these products will be $-\frac{h}{2}$, $\frac{h}{3}$, and so on: and the other terms will involve h^2, h^3, &c. Hence

$$y+k=a^x\{1+h(a-1-\tfrac{1}{2}\overline{a-1}^2+\tfrac{1}{3}\overline{a-1}^3-\&c)+Qh^2+\&c.)$$

$$=a^x+a^xAh+a^xQh^2+\&c.$$

Subtracting from this, $y=a^x$, and dividing by h,

$$\frac{k}{h}=a^xA+a^xQh+\&c.$$

And when the increment of x vanishes, this becomes

$$\frac{dy}{dx}=a^xA.$$

79. If $x=a^y$, y is the *logarithm* of x to the *base* a, and is denoted thus, $y=\log x$, or $y=\mathrm{l}x$.

80. If $y=\log x$, a the base, and A as before;

$$\frac{dy}{dx}=\frac{1}{Ax}.$$

Since $y=\log x$, $x=a^y$, and hence if y were the independent variable, $\frac{dx}{dy}=Aa^y=Ax$. Hence by Art. 45,

$$\frac{dy}{dx}=\frac{1}{Ax}.$$

81. If there be two expansions of the same function of x, according to powers of x: namely,

$$N + Px + Qx^2 + Rx^3 + \&c.,$$

$$\text{and} \quad n + px + qx^2 + rx^3 + \&c.,$$

$$\text{then} \quad N = n, \; P = p, \; Q = q, \; R = r, \; \&c.$$

For since the two series are expansions of the same function, they are equal for all values of x. Therefore their limits when x vanishes are equal; that is, $N = n$.

But taking away the equals N, n the remainders are equal. Dividing these by x, they become

$$P + Qx + Rx^2 + \&c.,$$

$$p + qx + rx^2 + \&c.,$$

which are equal for all values of x.

Therefore the limits are equal as before, or $P = p$.

Similarly taking away P and p, we prove Q and q equal; and so on.

82. To expand a^x according to powers of x.

$$\text{Let} \quad a^x = N + Px + Qx^2 + Rx^3 + \&c.$$

$$\text{Hence (Art. 78.)} \quad Aa^x = P + 2Qx + 3Rx^2 + \&c.$$

$$\text{But} \quad Aa^x = AN + APx + AQx^2 + \&c.$$

$$\text{Hence (Art. 81.)} \quad P = AN; \; 2Q = AP = A^2N, \; Q = \frac{A^2N}{2};$$

$$3R = AQ = \frac{A^3N}{2}, \; R = \frac{A^3N}{2 \, . \, 3};$$

and so on. Also making $x = 0$, $N = a^0 = 1$.

$$\text{Hence} \quad a^x = 1 + \frac{Ax}{1} + \frac{A^2x^2}{1 \, . \, 2} + \frac{A^3x^3}{1 \, . \, 2 \, . \, 3} + \&c.$$

83. Let e be the value of a, for which $A = 1$: then

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1 \,.\, 2} + \frac{x^3}{1 \,.\, 2 \,.\, 3} + \&\text{c.}; \text{ and making } x = 1,$$

$$e = 1 + 1 + \frac{1}{1 \,.\, 2} + \frac{1}{1 \,.\, 2 \,.\, 3} + \&\text{c.} = 2.71828, \&\text{c.}$$

But putting $\frac{1}{A}$ for x in the expansion of a^x, we have

$$a^{\frac{1}{A}} = 1 + 1 + \frac{1}{1 \,.\, 2} + \frac{1}{1 \,.\, 2 \,.\, 3} + \&\text{c.}$$

$$\text{Hence } e = a^{\frac{1}{A}}, \; e^A = a.$$

Hence (Art. 79) A is the logarithm of a to base e.

The logarithms to base e are called *natural* logarithms (formerly *hyperbolic*, or *Napierian* logarithms).

The natural logarithm of x may be denoted thus, $\mathrm{l}\, x$, the logarithm to any base a being denoted thus, $\log x$. Thus $A = \mathrm{l}\, a$.

84. The function $\log x$ cannot be expanded according to powers of x, with finite coefficients.

For if posible, let $\log x = N + Px + Qx^2 + \&\text{c.}$

And since this must be true for all values of x, let $x = 0$. Therefore $\log 0 = N$: therefore $a^N = 0$; which is only possible by supposing N to be an infinite negative quantity. Therefore the function cannot be expanded in this form.

85. To expand $\log(1 + x)$ according to powers of x.

Let $\log(1 + x) = N + Px + Qx^2 + Rx^3 + \&\text{c.}$

$$\text{Differentiating (Art. 80.) } \frac{1}{A(1 + x)} = P + 2Qx + 3Rx^2 + \&\text{c.}$$

$$\text{But } \frac{1}{A(1 + x)} = \frac{1}{A} - \frac{x}{A} + \frac{x^2}{A} - \&\text{c.}$$

Hence $P = \frac{1}{A}$; $2Q = -\frac{1}{A}$, $Q = -\frac{1}{2A}$;

$$3R = \frac{1}{A},\quad R = \frac{1}{3A};\ \text{\&c.}$$

Also in the original equation, making $x = 0$, $\log 1 = N$. Whence $1 = a^N$, which is only true if $N = 0$.

$$\text{Hence } \log(1 + x) = \frac{1}{A}\left\{\frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \text{\&c.}\right\}$$

If l represent the natural logarithms, since for these

$$A = 1,$$

$$l\,(1 + x) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \text{\&c.}$$

86. (Maclaurin's Theorem.) If y be any function of x, and if

$$y,\ \frac{dy}{dx},\ \frac{d^2y}{dx^2},\ \frac{d^3y}{dx^3},\ \text{\&c.}$$

become Y, Y_1, Y_2, Y_3, &c. by making $x = 0$; then

$$u = Y + \frac{Y_1}{1}x + \frac{Y_2}{1\,.\,2}x^2 + \frac{Y_3}{1\,.\,2\,.\,3}x^3 - \text{\&c.}$$

For let $y = N + Px + Qx^2 + Rx^3 + \text{\&c.}$;

$$\text{whence } \frac{dy}{dx} = P + 2Qx + 3Rx^2 + \text{\&c.}$$

$$\frac{d^2y}{dx^2} = 1\,.\,2\,Q + 2\,.\,3\,Rx + \text{\&c.}$$

$$\frac{d^3y}{dx^3} = 1\,.\,2\,.\,3\,R + \text{\&c.}$$

And making $x = 0$, in each of these equations

$$\begin{aligned} Y &= N, \\ Y_1 &= P, \\ Y_2 &= 1\,.\,2\,Q, \\ Y_3 &= 1\,.\,2\,.\,3\,R, \\ &\&\text{c.}; \end{aligned}$$

whence the proposition is manifest.

87. For example, to expand $\sin x$,

$$\frac{d.\sin x}{dx} = \cos x; \quad \frac{d^2 \sin x}{dx^2} = -\sin x, \quad \frac{d^3 \sin x}{dx^3} = -\cos x, \quad \frac{d^4 \sin x}{dx^4} = \sin x,$$

$\dfrac{d^5 \sin x}{dx^5} = \cos x$; and so on; whence, making $x = 0$.

$Y = 0$, $Y_1 = 1$, $Y_2 = 0$, $Y_3 = -1$, $Y_4 = 0$, $Y_5 = 1$; and so on.

And by the last Article,

$$\sin x = x - \frac{x^3}{1\,.\,2\,.\,3} + \frac{x^5}{1\,.\,2\,.\,3\,.\,4\,.\,5} - \&\text{c.}$$

88. In like manner we should find

$$\cos x = 1 - \frac{x^2}{1\,.\,2} + \frac{x^4}{1\,.\,2\,.\,3\,.\,4} - \&\text{c.}$$

89. If in Article 83, we put $x\sqrt{-1}$ for x, we have

$$e^{x\sqrt{-1}} = 1 + x\sqrt{-1} - \frac{x^2}{1.2} - \frac{x^3\sqrt{-1}}{1.2.3} + \frac{x^4}{1.2.3.4} + \frac{x^5\sqrt{-1}}{1.2.3.4.5} - \&\text{c.}$$

$= \cos x + \sqrt{-1} \sin x$, by the last two articles.

In the same manner $e^{-x\sqrt{-1}} = \cos x - \sqrt{-1} \sin x$.

90. Hence if $z = \cos x + \sqrt{-1} \sin x$, $\dfrac{1}{z} = \cos x - \sqrt{-1} \sin x$,

and $z + \dfrac{1}{z} = 2 \cos x$, $z - \dfrac{1}{z} = 2\sqrt{-1} \sin x$.

91. In the same manner,

$$e^{\pm mx\sqrt{-1}} = \cos mx \pm \sqrt{-1} \sin mx.$$

Hence z being as before, $z^m = \cos mx + \sqrt{-1} \sin mx$,

$$\frac{1}{z^m} = \cos mx - \sqrt{-1} \sin mx, \quad z^m + \frac{1}{z^m} = 2 \cos mx.$$

92. To resolve $z^{2m} - 2pz^m + 1$ into quadratic divisors, p being less than 1.

Let $p = \cos\theta$; and let $z^{2m} - 2\cos\theta \,.\, z^m + 1 = 0$; therefore $z^m + \frac{1}{z^m} = 2\cos\theta$. Hence if $\cos\theta = \cos mx$, z is the same quantity as in Articles 90, 91: wherefore $z + \frac{1}{z} = 2\cos x$, $z^2 - 2\cos x \,.\, z + 1 = 0$. And since the quadratic factor $z^2 - 2\cos x \,.\, z + 1$ vanishes for the same value of z as $z^{2m} - 2pz^m + 1$, the former is a divisor of the latter*. And this is the case if $\cos mx = \cos\theta$.

Since

$$\cos\theta = \cos \overline{2\pi - \theta} = \cos \overline{2\pi + \theta} = \cos \overline{4\pi - \theta} = \cos \overline{4\pi + \theta}, \text{ \&c.}$$

x has various values. We shall have $\cos mx = \cos\theta$, if $mx = 2n\pi \pm \theta$, n being any whole number;

$$\text{whence } x = \frac{2n\pi \pm \theta}{m}.$$

The two different signs $+$, $-$, give no different values of $\cos x$; but the different values of n, namely 0, 1, 2, as far as $m - 1$, give as many different values of $\cos x$, after which the values recur.

---

* Let $z^2 - 2\cos x \,.\, z + 1$ divide $z^{2m} - 2pz^m + 1$, with a quotient Q and a remainder R, which will be of the form $az + b$, as is easily seen. Then

$$z^{2m} - 2pz^m + 1 = Q\,(z^2 - 2\cos x \,.\, z + 1) + R.$$

Put for z the value which makes both the trinomials vanish, and we have $0 = 0 + R$; whence R is 0, and the quadratic factor divides $z^{2m} - 2pz^m + 1$ without a remainder.

For example, if $n = m + 1$,

$$\cos x = \cos\frac{\overline{2m+1}\;\pi \pm \theta}{m} = \cos\left(2\pi + \frac{2\pi \pm \theta}{m}\right) = \cos\frac{2\pi \pm \theta}{n};$$

which is the same value as when $n = 1$: and similarly for other values.

93. Example. To find the quadratic factors of

$$x^6 - x^3 + 1.$$

In this case $\cos\theta = \frac{1}{2}$, $\theta = \frac{\pi}{3}$, the values of x are $\frac{\pi}{9}$, $\frac{7\pi}{9}$, $\frac{13\pi}{9}$; and the quadratic factors are

$$x^2 - 2\cos\frac{\pi}{9}\,.\,x + 1,\;\; x^2 + 2\cos\frac{2\pi}{9}\,.\,x + 1,\;\; x^2 + 2\cos\frac{4\pi}{9}\,.\,x + 1.$$

94. To resolve $x^m - 1$ into quadratic factors. Squaring the expression, it becomes $x^{2m} - 2x^m + 1$, which has the same quadratic factors as the given expression, each recurring twice over. But the quadratic factors of $x^{2m} - 2x^m + 1$ may be found by Art. 92; θ being here $= 0$; since $\cos 0 = 1$. Hence the values of x are 0, $\frac{2\pi}{m}$, $\frac{4\pi}{m}$, and so on. And the quadratic factors are included in the formula

$$x^2 - 2\cos\frac{2n\pi}{m}\,.\,x + 1.$$

If m be even, n may have all values from 0 to $\frac{m}{2}$ inclusive. This gives two factors, $x^2 - 2x - 1$, and $x^2 + 2x + 1$; or $(x - 1)^2$ and $(x + 1)^2$; but in these, $x - 1$ and $x + 1$ only are factors of $x^m - 1$; and $x^2 - 1$ is the quadratic factor which contains these two.

If m be odd, n may have all values from 0 to $\frac{m-1}{2}$ inclusive. In this case the first and last quadratic expressions are $x^2 - 2x + 1$, and $x^2 - 2\cos\frac{m-1}{m}\pi . x + 1$; or $(x-1)^2$, and $x^2 + 2\cos\frac{\pi}{m} . x + 1$. Of these $x - 1$ occurs only as a simple factor.

95. EXAMPLES. To find the quadratic factors of $x^6 - 1$.

The quadratic factors are $x^2 - 1$, $x^2 - x + 1$, $x^2 + x + 1$.

For the values of $\cos x$ are $\cos\frac{2\pi}{6}$ and $\cos\frac{4\pi}{6}$, or $\cos\frac{\pi}{3}$ and $\cos\frac{2\pi}{3}$; which are $\frac{1}{2}$ and $-\frac{1}{2}$.

To find the quadratic factors of $x^5 - 1$.

The quadratic factors are

$$x - 1,\quad x^2 - 2\cos\frac{2\pi}{5} . x + 1,\quad x^2 + 2\cos\frac{\pi}{5} . x + 1.$$

For $\cos\frac{4\pi}{5} = -\cos\frac{\pi}{5}$.

96. In nearly the same manner we may shew that the general formula for the quadratic sectors of $x^m + 1$, is

$$x^2 - 2\cos\frac{\overline{2n+1}\,\pi}{m} . x + 1;$$

When m is even, n being taken from 0 to $\frac{m}{2}$ inclusive, gives two superabundant simple factors.

And when m is odd, n being taken from 0 to $\frac{m-1}{2}$ inclusive, gives one superabundant simple factor.

97. (TAYLOR'S THEOREM.) If y be any function of x, and if x become $x + h$, y becomes

$$y + \frac{dy}{dx} . \frac{h}{1} + \frac{d^2y}{dx^2} . \frac{h^2}{1.2} + \frac{d^3y}{dx^3} . \frac{h^3}{1.2.3} + \&c.$$

Let $x + h = x'$, and when x becomes x' let y become y'. Then by Art. 66, since y' is a function of h,

$$y' = Y + Y_1 h + Y_2 \frac{h^2}{1.2} + Y_3 \frac{h^3}{1.2.3} + \&c.$$

Where Y, Y_1, Y_2, Y_3, &c. are what y', $\frac{dy'}{dh^1}$, $\frac{d^2y'}{dh^2}$, $\frac{d^3y'}{dh^3}$, &c. become by making $h = 0$. Hence $Y = y$.

But since h only is supposed to vary in $x + h$ or x', it is evident that $\frac{dy'}{dh}$ is the same with $\frac{dy'}{dx'}$. And since y' is the same function of x' that y is of x, $\frac{dy'}{dx'}$ is the same function of x' or $x+h$, that $\frac{dy}{dx}$ is of x. Hence $\frac{dy'}{dx'}$, when h is 0, (that is, Y_1), is the same function of x which $\frac{dy}{dx}$ is of x: that is, $Y_1 = \frac{dy}{dx}$.

By exactly the same reasoning it would appear that

$$Y_2 = \frac{d^2y}{dx^2}, \quad Y_3 = \frac{d^3y}{dx^3}, \text{ \&c. Whence}$$

$$y' = y + \frac{dy}{dx}.h + \frac{d^2y}{dx^2}.\frac{h^2}{1.2} + \frac{d^3y}{dx^3}.\frac{h^3}{1.2.3} + \&c.$$

98. In the same manner if x become $x - h$, and y become $y_\backslash$,

$$y_\backslash = y - \frac{dy}{dx}.h + \frac{d^2y}{dx^2}.\frac{h^2}{1.2} - \frac{d^3y}{dx^3}.\frac{h^3}{1.2.3} + \&c.$$

99. If we have a series proceeding according to integral positive powers of h, the coefficients of all the terms being always finite; any term may be made greater than all those that follow by diminishing h.

Let $Nh^n + Ph^{n+1} + Qh^{n+2} + Rh^{n+3} + \&c.$ be any part of a series: and let S be the greatest of all the coefficients P, Q, R, &c. Therefore $P + Qh + Rh^2 + \&c.$ is less than $S + Sh + Sh^2 + \&c.$: that is, less than $\frac{S}{1-h}$.

Now Nh^n will be greater than all the terms which follow, namely, $Ph^{n+1} + Qh^{n+2} + Rh^{n+3} + \&c.$ if N be greater than $Ph + Qh^2 + Rh^3 + \&c.$; and still more, if N be greater than $Sh + Sh^2 + Sh^3 + \&c.$; that is, greater than $\frac{Sh}{1-h}$.

Therefore Nh^n will be greater than all the following terms:

if N be greater than $\frac{Sh}{1-h}$,

if $N - Nh$ be greater than Sh,

if h be less than $\frac{N}{N+S}$.

100. If y be a function of x, and if x have such a value that $\frac{dy}{dx} = 0$, the function y is a maximum or minimum according as $\frac{d^2y}{dx^2}$ is negative or positive.

Let x, y, be the values for which $\frac{dy}{dx} = 0$, and let $x - h$, $x + h$ be two other values, preceding and succeeding x. Then by Taylor's Theorem (Arts. 97, 98.), since $\frac{dy}{dx} = 0$,

$$y_{\prime} = y + \frac{d^2y}{dx^2}\cdot\frac{h^2}{1.2} - \frac{d^3y}{dx^3}\cdot\frac{h^3}{1.2.3} - \&c.$$

$$y' = y + \frac{d^2y}{dx^2}\cdot\frac{h^2}{1.2} + \frac{d^3y}{dx^3}\cdot\frac{h^3}{1.2.3} + \&c.$$

And by assuming h sufficiently small, the term involving h^2 in each of these series may be made greater than the sum of all which follow. Hence if $\frac{d^2y}{dx^2}$ be positive, both $y_\prime$, and y' (for $x-h$, $x+h$, and all intermediate values) will be greater than y; that is, y will be a minimum. And if $\frac{d^2y}{dx^2}$ be negative, both $y_\prime$, and y', and all intermediate values, will be less than y, and y will be a maximum.

101. If y be a function of x, and if the value of x which makes $\frac{dy}{dx}=0$, makes also $\frac{d^2y}{dx^2}=0$, y is neither a maximum nor a minimum; except this value make also $\frac{d^3y}{dx^3}=0$, in which case y is a maximum or minimum as $\frac{d^4y}{dx^4}$ is negative or positive.

For since $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ are both 0; by Arts. 97, 98,

$$y_\prime = y - \frac{d^3y}{dx^3}\cdot\frac{h^3}{1\,.\,2\,.\,3} + \&c.$$

$$y' = y + \frac{d^3y}{dx^3}\cdot\frac{h^3}{1\,.\,2\,.\,3} + \&c.$$

of which, when h is very small, one is greater and one less than y, whether $\frac{d^3y}{dx^3}$ be positive or negative.

But if $\frac{d^3y}{dx^3}=0$ also,

$$y_\prime = y + \frac{d^4y}{dx^4}\cdot\frac{h^4}{1\,.\,2\,.\,3\,.\,4} - \&c.$$

$$y' = y + \frac{d^4y}{dx^4}\cdot\frac{h^4}{1\,.\,2\,.\,3\,.\,4} + \&c.$$

which, when h is very small, are both greater than y when $\frac{d^4y}{dx^4}$ is positive, and both less than y when $\frac{d^4y}{dx^4}$ is negative.

And in like manner it may be shewn generally, that if several of the successive differential coefficients, beginning with the first, vanish; if the first which does not vanish be an odd one (as the third, fifth, &c.) the function will be neither a maximum nor a minimum. But if the first differential coefficient which does not vanish be an even one, the function will be a maximum if this coefficient is a negative quantity, and a minimum if it is a positive quantity.

102. At any point of a curve, the first and second differential coefficients have the same value in the curve, and in the circle of curvature.

In the diagram, p. 116, if $MN = h$, OR will be $\frac{dy}{dx}h$, both in the circle and in the curve.

Also, by Taylor's Theorem,

$$NQ = y + \frac{dy}{dx}.h + \frac{d^2y}{dx^2}.\frac{h^2}{1.2} + \frac{d^3y}{dx^3}.\frac{h^3}{1.2.3} + \&c.$$

and $NR = y + \dfrac{dy}{dx}.h$,

whence $QR = -\dfrac{d^2y}{dx^2}.\dfrac{h^2}{1.2} - \dfrac{d^3y}{dx^3}.\dfrac{h^3}{1.2.3} - \&c.$

But the circle of curvature ultimately coincides with the circle which touches PR and passes through Q. Hence the subtense of the circle of curvature at the point R is ultimately equal to QR. But if y' be the ordinate of the circle corresponding to the abscissa $x+h$, the subtense will be, by the same reasoning as above,

$$-\frac{d^2y'}{dx^2}.\frac{h^2}{1.2} - \frac{d^3y'}{dx^3}.\frac{h^3}{1.2.3} - \&c.$$

Hence these two series are equal; whence

$$\frac{d^2y'}{dx^2} + \frac{d^3y'}{dx^3} \cdot \frac{h}{3} + \&c. = \frac{d^2y}{dx^2} + \frac{d^3y}{dx^3} \cdot \frac{h}{3} + \&c.$$

And as this is true whatever be h, we must have

$$\frac{d^2y'}{dx^2} = \frac{d^2y}{dx^2}.$$

103. The curve is convex or concave to the axis, as $\frac{d^2y}{dx^2}$ is positive or negative.

For if the curve be convex, R is above Q, and RQ is positive, that is,

$$\frac{d^2y}{dx^2} \cdot \frac{h^2}{1 \,.\, 2} + \frac{d^3y}{dx^3} \cdot \frac{h^3}{1 \,.\, 2 \,.\, 3} + \&c.$$

is positive. And since this is true however small be h, $\frac{d^2y}{dx^2}$ is positive. And in like manner if the curve be concave to the axis, $\frac{d^2y}{dx^2}$ is negative.

104. Hence at a point of contrary flexure, $\frac{d^2y}{dx^2} = 0$; as has been proved already.

BOOK VI.

THE INTEGRAL CALCULUS.

THE RULES OF INTEGRATION.

1. If y, p, be functions of x, such that $\frac{dy}{dx} = p$, y is the *Integral* of p.

2. This relation is denoted thus;

$$y = \int_x p, \quad \text{or} \quad y = \int p\,dx.$$

3. The *Integral Calculus* is a system of rules and processes, the object of which is to find the integrals of any functions.

4. The fundamental rules of integration are derived from the rules of differentiation.

$$\text{If } y = \frac{x^{m+1}}{m+1}, \text{ we have } \frac{dy}{dx} = x^m;$$

$$\text{therefore } \int_x x^m = \frac{x^{m+1}}{m+1}.$$

5. If $y = \frac{1}{m\,.\,\overline{n+1}\,b}(a + bx^m)^{n+1}$; $\frac{dy}{dx} = (a + bx^m)^n x^{m-1}$;

$$\text{therefore } \int_x (a + bx^m)^n x^{m-1} = \frac{1}{m\,.\,\overline{n+1}\,b}(a + bx^m)^{n+1}.$$

6. If $y = -\cos x$, $\frac{dy}{dx} = \sin x$;

$$\text{therefore } \int_x \sin x = -\cos x.$$

7. If $y = \sin^{-1} x$, $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1 - x^2}}$;

$$\text{therefore } \int_x \frac{1}{\sqrt{1 - x^2}} = \sin^{-1} x,$$

$$\text{similarly } \int_x - \frac{1}{\sqrt{1 - x^2}} = \cos^{-1} x.$$

8. If $y = \tan^{-1} x$, $\dfrac{dy}{dx} = \dfrac{1}{1 + x^2}$;

$$\text{therefore } \int_x \frac{1}{1 + x^2} = \tan^{-1} x.$$

9. If $y = \mathrm{l}\, x$, $\dfrac{dy}{dx} = \dfrac{1}{x}$;

$$\text{therefore } \int_x \frac{1}{x} = \mathrm{l}\, x.$$

10. If $y = \dfrac{1}{\mathrm{l}\, a} \cdot a^x$, $\dfrac{dy}{dx} = a^x$;

$$\text{therefore } \int_x a^x = \frac{1}{\mathrm{l}\, a} a^x.$$

11. If $y = cp$, when p is any function of x,

$$\frac{dy}{dx} = c \frac{dp}{dx};$$

$$\text{therefore } \int_x c \frac{dp}{dx} = c \int_x \frac{dp}{dx}.$$

12. If $y = p + c$, when p is any function of x,

$$\frac{dy}{dx} = \frac{dp}{dx};$$

$$\text{therefore } \int_x \frac{dp}{dx} = p + c.$$

The quantity c is the *correction* of the integral, or the *arbitrary constant;* it is to be determined in each case from the circumstances of the problem.

13. EXAMPLES:

$$\int_x \sqrt{x} = \frac{2x^{\frac{3}{2}}}{3}, \text{ by Art. 4.}$$

$$\int_x \frac{1}{\sqrt{x}} = 2\sqrt{x}, \text{ by Art. 4.}$$

$$\int_x x\sqrt{1+x^2} = \frac{1}{3}(1+x^2)^{\frac{3}{2}}, \text{ by Art. 5.}$$

$$\int_x \frac{x}{(1+x^2)^{\frac{3}{2}}} = -\frac{1}{\sqrt{1+x^2}}, \text{ by Art. 5.}$$

14. If p be any function of x, it may be proved in like manner that

$$\int_x p^m \frac{dp}{dx} = \frac{p^{m+1}}{m+1}.$$

$$\int_x (a+bp^m)^n p^{m-1} \frac{dp}{dx} = \frac{1}{m\,.\,\overline{n+1}\,b}(a+bp^m)^{n+1}.$$

$$\int_x \sin p \,.\, \frac{dp}{dx} = -\cos p.$$

$$\int_x \frac{1}{\sqrt{(1-p^2)}}\,\frac{dp}{dx} = \sin^{-1}p; \quad \int_x -\frac{1}{\sqrt{(1-p^2)}}\,\frac{dp}{dx} = \cos^{-1}p.$$

$$\int_x \frac{1}{1+p^2}\,\frac{dp}{dx} = \tan^{-1}p.$$

$$\int_x \frac{1}{p}\,\frac{dp}{dx} = \text{l}\,p.$$

$$\int_x a^p \frac{dp}{dx} = \frac{1}{\text{l}\,a}\,a^p.$$

$$\int_x (p+q+\&\text{c.}) = \int_x p + \int_x q + \&\text{c.}$$

15. **Examples:**

$$\int_x (a-x)\sqrt{(2ax-x^2)} = \tfrac{1}{3}(2ax-x^2)^{\frac{3}{2}}.$$

$$\int_x \frac{(1+\sqrt{1-x^2})^{\frac{3}{2}}\,x}{\sqrt{1-x^2}} = -\frac{2}{5}(1+\sqrt{1-x^2})^{\frac{5}{2}}.$$

$$\int_x \frac{1}{x^2}\sin\frac{2\pi}{x} = \frac{1}{2\pi}\cos\frac{2\pi}{x}.$$

$$\int_x \frac{x}{\sqrt{1-x^4}} = \tfrac{1}{2}\sin^{-1}x^2.$$

$$\int_x \frac{x}{1+x^4} = \tfrac{1}{2}\tan^{-1}x^2.$$

$$\int_x \frac{1}{1+x} = \mathrm{l}\,(1+x).$$

$$\int_x \frac{x}{1+x^2} = \tfrac{1}{2}\,\mathrm{l}\,(1+x^2).$$

$$\int_x x\,a^{1+x^2} = \frac{1}{2\,\mathrm{l}\,a}\,a^{1+x^2}.$$

16. In some cases a transformation is requisite.

$$\int_x \frac{1}{a^2-x^2} = \frac{1}{2a}\int_x\left\{\frac{1}{a-x}+\frac{1}{a+x}\right\}$$

$$= \frac{1}{2a}\left\{-\mathrm{l}\,(a-x)+\mathrm{l}\,(a+x)\right\} = \frac{1}{2a}\,\mathrm{l}\,\frac{a+x}{a-x}.$$

$$\int_x \frac{1}{x^2-a^2} = \frac{1}{2a}\int_x\left\{\frac{1}{x-a}-\frac{1}{x+a}\right\}$$

$$= \frac{1}{2a}\left\{\mathrm{l}\,(x-a)-\mathrm{l}\,(x+a)\right\} = \frac{1}{2a}\,\mathrm{l}\,\frac{x-a}{x+a}.$$

$$\int_x \frac{1}{a^2+x^2} = \frac{1}{a}\cdot\int_x \frac{\frac{1}{a}}{1+\frac{x^2}{a^2}} = \frac{1}{a}\tan^{-1}\frac{x}{a}.$$

$$\int_x \frac{1}{\sqrt{(a^2+x^2)}} = \int_x \frac{\sqrt{(a^2+x^2)}+x}{\sqrt{(a^2+x^2)}\left\{x+\sqrt{(a^2+x^2)}\right\}}$$

$$= \int_x \frac{1+\dfrac{x}{\sqrt{(a^2+x^2)}}}{x+\sqrt{(a^2+x^2)}} = \mathrm{l}\left\{x+\sqrt{(a^2+x^2)}\right\}.$$

$$\int_x \frac{1}{\sqrt{(x^2-a^2)}} = \int_x \frac{\sqrt{(x^2-a^2)}+x}{\sqrt{(x^2-a^2)}\left\{x+\sqrt{(x^2-a^2)}\right\}}$$

$$= \int_x \frac{1+\dfrac{x}{\sqrt{(x^2-a^2)}}}{x+\sqrt{(x^2-a^2)}} = \mathrm{l}\left\{x+\sqrt{(x^2-a^2)}\right\}.$$

$$\int_x \frac{1}{\sqrt{(a^2-x^2)}} = \int_x \frac{1}{\sqrt{\left(1-\dfrac{x^2}{a^2}\right)}}\,\frac{1}{a} = \sin^{-1}\frac{x}{a}.$$

17. To find $\int_x \dfrac{1}{\sqrt{(2ax+x^2)}}$.

Let $x + a = z$, $2ax + x^2 = z^2 - a^2$, $\dfrac{dz}{dx} = 1$,

and $\int_x \dfrac{1}{\sqrt{(2ax+x^2)}} = \int_x \dfrac{1}{\sqrt{(z^2-a^2)}}\,\dfrac{dz}{dx}$

$= \mathrm{l}\left\{z+\sqrt{(z^2-a^2)}\right\}$, by last Article,

$= \mathrm{l}\left\{x+a+\sqrt{(2ax+x^2)}\right\}$.

In like manner we may find

$$\int_x \frac{1}{\sqrt{(x^2-2ax)}} = \mathrm{l}\left\{x-a+\sqrt{(x^2-2ax)}\right\}.$$

Also $\int_x \dfrac{1}{\sqrt{(2ax-x^2)}}$, if $a - x = z$,

becomes $\int_x -\dfrac{1}{\sqrt{(a^2-z^2)}}\,\dfrac{dz}{dx} = \cos^{-1}\dfrac{z}{a}$.

But $\cos^{-1}\frac{z}{a} = \cos^{-1}\frac{a-x}{a} = \cos^{-1}\left(1-\frac{x}{a}\right)$, and since the cosine is $1-\frac{x}{a}$ the versed sine is $\frac{x}{a}$.

18. To find $\int_x \frac{1}{x\sqrt{(1+x^2)}}$.

$$\int_x \frac{1}{x\sqrt{(1+x^2)}} = \int_x \frac{1}{\sqrt{\left(\frac{1}{x^2}+1\right)}}\,\frac{1}{x^2}\text{: and if } \frac{1}{x} = z$$

$$= \int_x -\frac{1}{\sqrt{(z^2+1)}}\,\frac{dz}{dx} = -\mathbf{l}\,\{z+\sqrt{(z^2+1)}\}$$

$$= -\mathbf{l}\,\frac{1+\sqrt{(1+x^2)}}{x} = \mathbf{l}\,\frac{x}{1+\sqrt{(1+x^2)}}.$$

Similarly $\int_x \frac{1}{x\sqrt{(1-x^2)}} = \mathbf{l}\,\frac{x}{1+\sqrt{(1-x^2)}}$.

And $\int_x \frac{1}{x\sqrt{(x^2-1)}}$, if $\frac{1}{x} = z$,

becomes $\int_x -\frac{1}{\sqrt{(1-z^2)}}\,\frac{dz}{dx} = -\cos^{-1} z = -\cos^{-1}\frac{1}{x}$

$$= -\sec^{-1} x.$$

19. To find $\int_x \frac{1}{\sqrt{(1+2mx+x^2)}}$.

Let $x + m = z$, $1 + 2mx + x^2 = z^2 + 1 - m^2$.

$$\int_x \frac{1}{\sqrt{(1+2mx+x^2)}} = \int_x \frac{1}{\sqrt{(z^2+1-m^2)}}\,\frac{dz}{dx}$$

$$= \mathbf{l}\,\{z+\sqrt{(z^2+1-m^2)}\} = \mathbf{l}\,\{x+m+\sqrt{(1+2mx+x^2)}\}.$$

In nearly the same manner may be found

$$\int_x \frac{1}{\sqrt{(a + bx + cx^2)}},$$

a, b, c, being positive or negative.

20. To find $\int_x \frac{1}{x\sqrt{(a + bx + cx^2)}}$.

$$\int_x \frac{1}{x\sqrt{(a + bx + cx^2)}} = \int_x \frac{1}{\sqrt{a}\sqrt{\left(\frac{1}{x^2} + \frac{b}{ax} + \frac{c}{a}\right)}} \cdot \frac{1}{x^2}.$$

Let $\frac{1}{x} + \frac{b}{2a} = z$:

then $\frac{1}{x^2} = -\frac{dz}{dx}$: $\frac{1}{x^2} + \frac{b}{ax} + \frac{c}{a} = z^2 - \frac{b^2}{4a^2} + \frac{c}{a}$.

Hence the integral becomes

$$\int_x - \frac{1}{\sqrt{a}\sqrt{\left(z^2 - \frac{b^2 - 4ac}{4a^2}\right)}} \frac{dz}{dx},$$

which may be found by Art. 16.

INTEGRATION OF RATIONAL ALGEBRAICAL FRACTIONS.

21. By rational algebraical fractions are meant those in which the numerators and denominators consist of integral powers of the independent variable. They are integrated by resolving them into partial fractions, according to the factors of the denominator.

22. Let the factors of the denominator be all unequal.

If $u = \frac{1}{(x - a)(x - b)(x - c)}$, to find $\int_x u$.

Resolve u into partial fractions as follows:

$$\text{Let } u = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}.$$

Hence

$$1 = A(x-b)(x-c) + B(x-a)(x-c) + C(x-a)(x-b).$$

Put a for x: therefore $1 = A(a-b)(a-c)$.

$$\text{Hence } A = \frac{1}{(a-b)(a-c)};$$

$$\text{similarly } B = -\frac{1}{(a-b)(b-c)},\ C = \frac{1}{(a-c)(b-c)}$$

$$\int_x u = A\,\mathrm{l}(x-a) + B\,\mathrm{l}(x-b) + C\,\mathrm{l}(x-c).$$

23. Let m factors of the denominator be equal.

$$\text{If } u = \frac{1}{(x-a)^m(x-b)},\ \text{to find } \int_x u.$$

$$\text{Let } u = \frac{A}{(x-a)^m} + \frac{A_1}{(x-a)^{m-1}} + \frac{A_2}{(x-a)^{m-2}} + \&\text{c.} + \frac{B}{x-b},$$

$$1 = \{A + A_1(x-a) + A_2(x-a)^2 + \&\text{c.}\}(x-b) + B(x-a)^m.$$

Differentiating

$$0 = \{A_1 + 2A_2(x-a) + \&\text{c.}\}(x-b) + \{A + A_1(x-a) + A_2(x-a)^2 + \&\text{c.}\} + mB(x-a)^{m-1}.$$

Differentiating again

$$0 = \{1.2A_2 + \&\text{c.}\}(x-b) + 2\{A_1 + 2A_2(x-a) + \&\text{c.}\} + m.(m-1)B(x-a)^{m-2},$$

and so on.

Make $x = a$ in each of these equations; and we find

$$1 = A(a-b),$$
$$0 = A_1(a-b) + A,$$
$$0 = 1.2A_2(a-b) + 2A_1,$$
$$\&\text{c.} = \&\text{c.}$$

Hence $A = \dfrac{1}{a-b}$; $A_1 = -\dfrac{A}{a-b} = -\dfrac{1}{(a-b)^2}$;

$$A_2 = -\frac{A_1}{a-b} = \frac{1}{(a-b)^3},$$

and so on.

And $\int_x u = -\dfrac{A}{m-1}\,\dfrac{1}{(x-a)^{m-1}} - \dfrac{A_1}{m-2}\,\dfrac{1}{(x-a)^{m-2}}$

$$-\frac{A_2}{m-3}\,\frac{1}{(x-a)^{m-3}} - \text{\&c.} + B\,l(x-b).$$

24. Let the denominator have two imaginary simple factors.

If $u = \dfrac{1}{(x^2+a^2)(x-b)}$, to find $\int_x u$.

Let $u = \dfrac{Ax + A_1}{x^2+a^2} + \dfrac{B}{x-b}$,

$$1 = (Ax + A_1)(x-b) + B(x^2+a^2),$$

$$1 = Ax^2 + (Ab - A_1)x - A_1 b + B(x^2+a^2).$$

Let $x^2 + a^2 = 0$, whence $x = a\sqrt{-1}$,

$$1 = Aa^2 + (Ab - A_1)a\sqrt{-1} - A_1 b.$$

Hence, equating impossible and possible points,

$$0 = Ab - A_1,$$

$$1 = Aa^2 - A_1 b.$$

Hence $1 = Aa^2 - Ab^2$; $A = \dfrac{1}{a^2-b^2}$, $A_1 = \dfrac{b}{a^2-b^2}$.

Also $B = -\dfrac{1}{a^2-b^2}$.

Hence $u = \dfrac{1}{a^2-b^2}\left(\dfrac{x+b}{x^2+a^2} - \dfrac{1}{x-b}\right)$,

$$\int_x u = \frac{1}{a^2 - b^2}\left\{\tfrac{1}{2}\,\mathrm{l}\,(x^2 + a^2) + \frac{b}{a}\tan^{-1}\frac{x}{a} - \mathrm{l}\,(x - b)\right\}$$

$$= \frac{1}{a^2 - b^2}\left\{\frac{b}{a}\tan^{-1}\frac{x}{a} + \mathrm{l}\,\frac{\sqrt{(x^2 + a^2)}}{x - b}\right\}.$$

25. Let the denominator have several imaginary simple factors.

If $u = \dfrac{x^r}{x^{2m} - 2px^m + 1}$, to find $\int_x u$.

To resolve $\dfrac{x^r}{x^{2m} - 2px^m + 1}$ into partial fractions.

Let $p = \cos\theta$, and $\phi = \dfrac{2n\pi + \theta}{m}$, where n is any whole number; then $x^2 - 2\cos\phi\,.\,x + 1$ is a divisor of $x^{2m} - 2px^m + 1$.

Let $x^{2m} - 2px^m + 1 = Q\,(x^2 - 2\cos\phi\,.\,x + 1)$; and let

$$\frac{x^r}{x^{2m} - 2px^m + 1} = \frac{A\,(x - \cos\phi) + B}{x^2 - 2\cos\phi\,.\,x + 1} + \frac{P}{Q}.$$

Hence $x^r = \{A\,(x - \cos\phi) + B\}\,Q + P\,\{x^2 - 2\cos\phi\,.\,x + 1\}$.

Let $x^2 - 2\cos\phi\,.\,x + 1 = 0$, whence $x = \cos\phi + \sqrt{-1}\,\sin\phi$; call this x'. And let Q' be the value of Q which arises from putting x' for x.

$$\text{Then } (A\sqrt{-1}\,\sin\phi + B)\,Q' = x'^r.$$

$$\text{But } Q\,(x^2 - 2\cos\phi\,.\,x + 1) = x^{2m} - 2px^m + 1.$$

Differentiate

$$Q\,(2x - 2\cos\phi) + \frac{dQ}{dx}(x^2 - 2\cos\phi\,.\,x + 1) = 2mx^{2m-1} - 2mpx^{m-1}.$$

Put x' for x; and we have

$$Q'\,(2x' - 2\cos\phi) = 2mx'^{2m-1} - 2mpx'^{m-1},$$

$$Q' = mx'^{m-1}\,\frac{x'^m - p}{x' - \cos\phi}.$$

But $x'^m - p = \sqrt{-1}\sqrt{1-p^2}$, $x' - \cos\phi = \sqrt{-1}\sin\phi$,

$$Q' = mx'^{m-1}\frac{\sqrt{1-p^2}}{\sin\phi}.$$

Hence $(A\sqrt{-1}\sin\phi + B)\,mx'^{m-1}\dfrac{\sqrt{1-p^2}}{\sin\phi} = x'^r$,

$$A\sqrt{-1}\sin\phi + B = \frac{\sin\phi}{m\sqrt{1-p^2}}x'^{-m+r+1}.$$

But (Book v. Art. 91)

$$x'^{-m+r+1} = x'^{-(m-r-1)} = \cos(m-r-1)\phi - \sqrt{-1}\sin(m-r-1)\phi.$$

Substituting this and equating possible and impossible parts,

$$A = -\frac{\sin(m-r-1)\phi}{m\sqrt{1-p^2}};\quad B = \frac{\sin\phi\,.\cos(m-r-1)\phi}{m\sqrt{1-p^2}};$$

$$\int_x u = \frac{A}{2}\,\mathrm{l}\,(x^2 - 2\cos\phi\,.\,x + 1) + \frac{B}{\sin\phi}\tan^{-1}\frac{x-\cos\phi}{\sin\phi} + \&c.;$$

or $\int_x u = A\,\mathrm{l}\sqrt{(x^2 - 2\cos\phi\,.\,x+1)}$

$$+\frac{B}{\sin\phi}\sec^{-1}\frac{\sqrt{(x^2-2\cos\phi\,.\,x+1)}}{\sin\phi} + \&c.$$

and there will be similar terms for each of the different values of ϕ.

26. Let $u = \dfrac{1}{(x^m - 1)\,x^r}$, to find $\int_x u$.

Let $\phi = \dfrac{2n\pi}{m}$, then $x^2 - 2\cos\phi\,.\,x + 1$ is a factor of $x^m - 1$,

Hence let $(x^2 - 2\cos\phi\,.\,x + 1)\,Q = (x^m - 1)\,x^r = x^{m+r} - x^r$, and let $u = \dfrac{A(x - \cos\phi) + B}{x^2 - 2\cos\phi . x + 1} + \dfrac{P}{Q}$; whence reducing

$$1 = \{A(x - \cos\phi) + B\} Q + P(x^2 - 2\cos\phi . x + 1).$$

For x put $x' = \cos\phi + \sqrt{-1}\sin\phi$, and let Q' be what Q becomes by this substitution: hence

$$1 = \{A\sqrt{-1}\sin\phi + B\} Q'.$$

To find Q', differentiate the original assumption in Q, and we have

$$2(x - \cos\phi) Q + (x^2 - 2\cos\phi . x + 1)\frac{dQ}{dx} = \overline{m + r}\, x^{m+r-1} - r x^{r-1}.$$

For x put x', and we have

$$2\sqrt{-1}\sin\phi . Q' = \overline{m + r}\, x'^{m+r-1} - r x'^{r-1},$$

or $$2\sqrt{-1}\sin\phi . Q' = \overline{m + r}\cos\overline{m + r - 1}\,\phi - r\cos\overline{r - 1}\,\phi$$
$$+ \sqrt{-1}\,(\overline{m + r}\sin\overline{m + r - 1}\,\phi - r\sin\overline{r - 1}\,\phi).$$

But by what precedes,

$$2\sqrt{-1}\sin\phi = (A\sqrt{-1}\sin\phi + B)\, 2\sqrt{-1}\sin\phi . Q';$$

whence

$$2\sqrt{-1}\sin\phi = A\sqrt{-1}\sin\phi\,(\overline{m + r}\cos\overline{m + r - 1}\,\phi - \cos\overline{r - 1}\,\phi)$$
$$- A\sin\phi\,(\overline{m + r}\sin\overline{m + r - 1}\,\phi - r\sin\overline{r - 1}\,\phi)$$
$$+ B\,(\overline{m + r}\cos\overline{m + r - 1}\,\phi - r\cos\overline{r - 1}\,\phi)$$
$$+ B\sqrt{-1}\,(\overline{m + r}\sin\overline{m + r - 1}\,\phi - r\sin\overline{r - 1}\,\phi).$$

Equating possible parts, we find

$$B = A\sin\phi\,\frac{\overline{m + r}\sin\overline{m + r - 1}\,\phi - r\sin\overline{r - 1}\,\phi}{\overline{m + r}\cos\overline{m + r - 1}\,\phi - r\cos\overline{r - 1}\,\phi}.$$

Substituting and equating impossible parts, we find

$$2\sin\phi = A\sin\phi\left\{\overline{m+r}\cos\overline{m+r-1}\,\phi - r\cos\overline{r-1}\,\phi\right.$$

$$\left.+\frac{(\overline{m+r}\sin\overline{m+r-1}\,\phi - r\sin\overline{r-1}\,\phi)^2}{\overline{m+r}\cos\overline{m+r-1}\,\phi - r\sin\overline{r-1}\,\phi}\right\}$$

$$= A\sin\phi\,\frac{\overline{m+r}\rceil^2 - 2\,\overline{mr+r^2}\cos m\phi + r^2}{\overline{m+r}\cos\overline{m+r-1}\,\phi - r\sin\overline{r-1}\,\phi}.$$

But $\cos m\phi = \cos 2n\pi = 1$; hence the numerator of the fraction is m^2;

$$\text{and } A = \frac{2}{m^2}(\overline{m+r}\cos\overline{m+r-1}\,\phi - r\cos\overline{r-1}\,\phi),$$

$$B = \frac{2\sin\phi}{m^2}(\overline{m+r}\sin\overline{m+r-1}\,\phi - r\sin\overline{r-1}\,\phi).$$

Whence $\int_x u$ may be found as in last Article.

REDUCTION OF BINOMIAL INTEGRALS.

The integrals here spoken of are of the form

$$\int_x (a+bx^n)^p\,x^{m-1}.$$

They are found by reducing the indices m and p.

27. To find $\int_x (a^2+x^2)^{\frac{1}{2}}\,x^7$.

Assume

$$P_{n-1} = (a^2+x^2)^{\frac{3}{2}}\,x^{n-1},\quad U_n = \int_x (a^2+x^2)^{\frac{1}{2}}\,x^n$$

$$\frac{dP_{n-1}}{dx} = (a^2+x^2)^{\frac{1}{2}}\,3x^n + (a^2+x^2)^{\frac{3}{2}}\,(n-1)\,x^{n-2}$$

$$= (a^2+x^2)^{\frac{1}{2}}\,3x^n + a^2(a^2+x^2)^{\frac{1}{2}}(n-1)\,x^{n-2} + (a^2+x^2)^{\frac{1}{2}}(n-1)x^n$$

$$= (n+2)\,(a^2+x^2)^{\frac{1}{2}}\,x^n + (n-1)\,a^2\,(a^2+x^2)^{\frac{1}{2}}\,x^{n-2}.$$

Integrating,

$$P_{n-1} = (n+2)\,U_n + (n-1)\,a^2\,U_{n-2},$$

$$U_n = \frac{1}{n+2}\,P_{n-1} - \frac{(n-1)\,a^2}{n+2}\,U_{n-2}.$$

Hence putting values for n,

$$U_7 = \frac{1}{9} P_6 - \frac{6a^2}{9} U_5,$$

$$U_5 = \frac{1}{7} P_4 - \frac{4a^2}{7} U_3,$$

$$U_3 = \frac{1}{5} P_2 - \frac{2a^2}{5} U_1,$$

$$U_7 = \frac{1}{9} P_6 - \frac{6a^2}{9.7} P_4 + \frac{6 . 4a^4}{9.7.5} P_2 - \frac{6.4.2.a^6}{9.7.5} U_1.$$

Also

$$U_1 = \int_x (a^2 + x^2)^{\frac{1}{2}} x = \tfrac{1}{3} (a^2 + x^2)^{\frac{3}{2}}.$$

Hence

$$U_7 = (a^2 + x^2)^{\frac{3}{2}} \left\{ \frac{x^6}{7} - \frac{6a^2x^4}{9.7} + \frac{6.4a^4x^2}{9.7.5} - \frac{6.4.2a^6}{9.7.5.3} \right\}.$$

28. To find $\int_x (x^2 - a^2)^{\frac{9}{2}}$.

Let $U_n = \int_x (x^2 - a^2)^n$. Assume $P_n = (x^2 - a^2)^n x$.

$$\frac{dP_n}{dx} = (x^2 - a^2)^n + 2n (x^2 - a^2)^{n-1} x^2$$

$$= (x^2 - a^2)^n + 2n (x^2 - a^2)^n + 2na^2 (x^2 - a^2)^{n-1}$$

$$= (2n + 1) (x^2 - a^2)^n + 2na^2 (x^2 - a^2)^{n-1}.$$

Integrating,

$$P_n = (2n + 1) U_n + 2na^2 U_{n-1}.$$

Hence

$$U_n = \frac{1}{2n + 1} P_n - \frac{2na^2}{2n + 1} U_{n-1}.$$

Therefore

$$U_{\frac{9}{2}} = \frac{1}{10} P_{\frac{9}{2}} - \frac{9a^2}{10} U_{\frac{7}{2}},$$

$$U_{\frac{7}{2}} = \frac{1}{8} P_{\frac{7}{2}} - \frac{7a^2}{8} U_{\frac{5}{2}},$$

$$U_{\frac{5}{2}} = \frac{1}{6} P_{\frac{5}{2}} - \frac{5a^2}{6} U_{\frac{3}{2}},$$

$$U_{\frac{3}{2}} = \frac{1}{4} P_{\frac{3}{2}} - \frac{3a^2}{4} U_{\frac{1}{2}},$$

$$U_{\frac{1}{2}} = \frac{1}{2} P_{\frac{1}{2}} - \frac{a^2}{2} U_{-\frac{1}{2}}.$$

Hence

$$U_{\frac{9}{2}} = \frac{1}{10} P_{\frac{9}{2}} - \frac{9a^2}{10 . 8} P_{\frac{7}{2}} + \frac{9 \cdot 7 a^4}{10 . 8 . 6} P_{\frac{5}{2}} - \frac{9 \cdot 7 \cdot 5 a^6}{10 . 8 . 6 . 4} P_{\frac{3}{2}}$$

$$+ \frac{9 \cdot 7 \cdot 5 \cdot 3 a^8}{10 . 8 . 6 . 4 . 2} P_{\frac{1}{2}} - \frac{9 \cdot 7 \cdot 5 \cdot 3 a^{10}}{10 . 8 . 6 . 4 . 2} U_{-\frac{1}{2}}.$$

But

$$U_{-\frac{1}{2}} = \int_x \frac{1}{(x^2 - a^2)^{\frac{1}{2}}} = \text{l}\,(x + \sqrt{x^2 - a^2}).$$

Hence the integral is known.

29. To find $\int_x \frac{1}{(1 + x^2)^m x^2}$. Suppose this $= U_m$.

Assume $P_{m-1} = \frac{1}{(1 + x^2)^{m-1} x}$.

Hence
$$\frac{dP_{m-1}}{dx} = - \frac{1}{(1 + x^2)^{m-1} x^2} - \frac{2(m-1)}{(1 + x^2)^m}$$

$$= - \frac{1}{(1 + x^2)^{m-1} x^2} - \frac{2(m-1)(1 + x^2 - 1)}{(1 + x^2)^m x^2}$$

$$= - \frac{(2m-1)}{(1 + x^2)^{m-1} x^2} + \frac{2m-2}{(1 + x^2)^m x^2}.$$

Integrating, $P_{m-1} = -(2m-1)\,U_{m-1} + (2m-2)\,U_m$

$$U_m = \frac{1}{2m-2}P_{m-1} + \frac{2m-1}{2m-2}U_{m-1}.$$

We may by this formula reduce the integral to U_1.

$$\text{But}\quad U_1 = \int_x \frac{1}{(1+x^2)\,x^2} = \int_x \left\{\frac{1}{x^2} - \frac{1}{1+x^2}\right\}$$

$$= -\frac{1}{x} - \tan^{-1} x:$$

Whence U_m is found.

30. To find $\displaystyle\int_x \frac{1}{x^{10}\sqrt{(1+x^3)}}$.

$$\text{Suppose}\quad \int_x \frac{1}{x^m\sqrt{(1+x^3)}} = U_m.$$

$$\text{Assume}\quad P_{m-1} = \frac{\sqrt{(1+x^3)}}{x^{m-1}}.$$

$$\frac{dP_{m-1}}{dx} = \frac{3}{2x^{m-3}\sqrt{(1+x^3)}} - \frac{(m-1)\sqrt{(1+x^3)}}{x^m}$$

$$= \frac{3}{2x^{m-3}\sqrt{(1+x^3)}} - \frac{m-1}{x^m\sqrt{(1+x^3)}} - \frac{m-1}{x^{m-3}\sqrt{(1+x^3)}}$$

$$= -\frac{m-1}{x^m\sqrt{(1+x^3)}} - \frac{2m-5}{2x^{m-3}\sqrt{(1+x^3)}}.$$

Integrating, $P_{m-1} = -(m-1)\,U_m - \dfrac{2m-5}{2}U_{m-3}$;

$$U_m = -\frac{1}{m-1}P_{m-1} - \frac{2m-5}{2m-2}U_{m-3};$$

$$\text{Hence,}\quad U_{10} = -\frac{1}{9} P_9 - \frac{15}{18} U_7;$$

$$U_7 = -\frac{1}{6} P_6 - \frac{9}{12} U_4;$$

$$U_4 = -\frac{1}{3} P_3 - \frac{3}{6} U_1;$$

Hence the integral is reduced to U_1, or $\int_x \frac{1}{x\sqrt{(1+x^3)}}$.

31. The rule for the assumption in such cases as those of the four preceding articles may be thus stated.

To reduce the integral $\int_x (a+bx^n)^p x^{m-1}$.

Assume $P = (a+bx^n)^{p+1} x^m$ in all cases; except

(1) If m is to be reduced *down*, assume

$$P_{m-n} = (a+bx^n)^{p+1} x^{m-n}.$$

(2) If p is to be reduced *down*, assume

$$P_p = (a+bx^n)^p x^m.$$

The index is reduced *down*, when being positive it is diminished or being negative it is increased. It is reduced *up* in the contrary cases, as n in Arts. 27 and 28.

32. Such integrals may also be found by substitutions which make them rational.

To find $\int_x \frac{1}{x\sqrt[3]{(1+x^3)}}$. Assume $\sqrt[3]{(1+x^3)} = z$.

$$\text{Hence,}\quad x^3 = z^3 - 1,\quad \frac{3}{x} = \frac{3z^2}{z^3-1}\cdot\frac{dz}{dx}$$

$$\int_x \frac{1}{x\sqrt[3]{(1+x^3)}} = \int_x \frac{z}{z^3-1}\cdot\frac{dz}{dx}.$$

But $\frac{z}{z^3-1} = \frac{1}{6}\left\{\frac{2}{z-1} - \frac{2z+1}{z^2+z+1} + \frac{3}{z^2+z+1}\right\}$;

whence, $\int_z \frac{z}{z^3-1}\cdot\frac{dz}{dx}$

$$= \frac{1}{6}\left\{2\,l\,(z-1) - l\,(z^2+z+1) + \frac{3}{\sqrt{\frac{3}{4}}}\tan^{-1}\frac{z+\frac{1}{2}}{\sqrt{\frac{3}{4}}}\right\}$$

$$= \frac{1}{6}\,l\,\frac{(z-1)^3}{z^3-1} + \frac{1}{\sqrt{3}}\tan^{-1}\frac{2z+1}{\sqrt{3}}.$$

And $\int_x \frac{1}{x\sqrt[3]{(1+x^3)}}$

$$= \tfrac{1}{2}\,l\,\frac{\sqrt[3]{(1+x^3)}-1}{x} + \frac{1}{\sqrt{3}}\tan^{-1}\frac{2\sqrt[3]{(1+x^3)}+1}{\sqrt{3}}.$$

33. To find $\int_x \frac{x^3}{\sqrt[3]{(1+x^3)}}$.

Assume $\frac{\sqrt[3]{(1+x^3)}}{x} = z$.

Hence, $1+x^3 = z^3x^3$, $x^3 = \frac{1}{z^3-1}$, $3x^2 = \frac{3z^2}{(z^3-1)^2}\cdot\frac{dz}{dx}$;

$$\int_x \frac{x^3}{\sqrt[3]{(1+x^3)}} = \int_x \frac{x^2}{\frac{\sqrt[3]{(1+x^3)}}{x}} = \int_x \frac{z}{(z^3-1)^2}\cdot\frac{dz}{dx};$$

which is rational, and may be reduced by preceding articles.

EXAMPLES OF INTEGRALS.

34. To find the area of a parabola in which $y^{m+n} = a^m x^n$,

$$\text{area} = \int_x y = \int_x a^{\frac{m}{m+1}}x^{\frac{m}{m+n}} = \frac{m+n}{m+2n}a^{\frac{m}{m+n}}x^{\frac{m+2n}{m+n}} + C.$$

And if when $x = 0$, area $= 0$, $C = 0$;

$$\text{area} = \frac{m+n}{m+2n} a^{\frac{m}{m+n}} x^{\frac{m+2n}{m+n}} = \frac{m+n}{m+2n} yx.$$

35. To find the area of a half segment of a circle,

$$\text{area} = -\int_x y = -\int_x \sqrt{(a^2 - x^2)},$$

x being measured from the centre.

$$\text{Let } P = x\sqrt{(a^2 - x^2)}, \quad \frac{dP}{dx} = \sqrt{(a^2 - x^2)} - \frac{x^2}{\sqrt{(a^2 - x^2)}}$$

$$= \sqrt{(a^2 - x^2)} + \frac{a^2 - x^2}{\sqrt{(a^2 - x^2}} - \frac{a^2}{\sqrt{(a^2 - x^2)}}$$

$$= 2\sqrt{(a^2 - x^2)} - \frac{a^2}{\sqrt{(a^2 - x^2)}}.$$

$$\text{Hence } \sqrt{(a^2 - x^2)} = \frac{1}{2} \cdot \frac{dP}{dx} + \frac{a^2}{2\sqrt{(a^2 - x^2)}}$$

$$-\int_x \sqrt{(a^2 - x^2)} = -\frac{1}{2} \cdot P + \frac{a^2}{2}\cos^{-1}\frac{x}{a} + C$$

$$= -\frac{1}{2} \cdot x\sqrt{(a^2 - x^2)} + \frac{a^2}{2}\cos^{-1}\frac{x}{a} + C.$$

And the area of the segment is 0 when $x = a$.

Hence $0 = C$;

$$\text{area} = \frac{a^2}{2}\cos^{-1}\frac{x}{a} - \frac{x\sqrt{(a^2 - x^2)}}{2}.$$

36. To find the solid content of a segment of a sphere,

$$\text{solid content} = -\int_x \pi y^2 = -\pi\int_x (a^2 - x^2) = -\pi\left(a^2 x - \frac{x^3}{3}\right) + C;$$

and when $x = a$, the content $= 0$. Hence $C = \frac{2\pi a^3}{3}$,

$$\text{solid content} = \frac{\pi}{3}\left\{2a^3 - 3a^2x + x^3\right\}.$$

37. To find the length of the arc of the semi-cubical parabola.

$$\text{Here } y^3 = ax^2, \quad y = a^{\frac{1}{3}}x^{\frac{2}{3}}, \quad \frac{dy}{dx} = \frac{2}{3}\cdot\frac{a^{\frac{1}{3}}}{x^{\frac{1}{3}}},$$

$$\text{suppose arc} = \int_x \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} = \int_x \sqrt{\left(1 + \frac{4a^{\frac{2}{3}}}{9x^{\frac{2}{3}}}\right)}$$

$$= \int_x \sqrt{\left(x^{\frac{2}{3}} + \frac{4}{9}a^{\frac{2}{3}}\right)}\, x^{-\frac{1}{3}}.$$

This may be integrated as in Article 5, and gives

$$\text{arc} = \left(x^{\frac{2}{3}} + \frac{4}{9}a^{\frac{2}{3}}\right)^{\frac{3}{2}} + C.$$

And if the arc begins when $x = 0$,

$$\text{arc} = \left(x^{\frac{2}{3}} + \frac{4}{9}a^{\frac{2}{3}}\right)^{\frac{3}{2}} - \frac{8a}{27}.$$

38. To find the surface of the common parabolic conoid.

$$y^2 = 4ax, \quad y = 2\sqrt{ax}, \quad \frac{dy}{dx} = \frac{\sqrt{a}}{\sqrt{x}},$$

$$\text{surface} = \int_x 2\pi y \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} = \int_x 4\pi\sqrt{ax}\sqrt{\left(1 + \frac{a}{x}\right)}$$

$$= 4\pi\sqrt{a}\int_x \sqrt{(x+a)} = \frac{8\pi a}{3}(x+a)^{\frac{3}{2}} + C.$$

And if the surface begins when $x = 0$,

$$\text{surface} = \frac{8\pi a}{3}\left\{(x+a)^{\frac{3}{2}} - a^{\frac{3}{2}}\right\}.$$

39. When a body falls to a centre of force varying inversely as the square of the distance, to find the velocity and time.

By B. v. Art. 71, $-v\dfrac{dv}{dx}=f$; and $f=\dfrac{\mu}{x^2}$,

$$v\frac{dv}{dx}=-\frac{\mu}{x^2},\quad \frac{v^2}{2}=\frac{\mu}{x}+C.$$

If h be the original distance fallen from, $v=0$, when $x=h$.

$$\frac{v^2}{2}=\frac{\mu}{x}-\frac{\mu}{h};\quad v^2=2\mu\left\{\frac{1}{x}-\frac{1}{h}\right\},\quad v=\sqrt{2\mu}\,.\sqrt{\frac{h-x}{hx}}.$$

For the time, $\dfrac{dt}{dx}=-\dfrac{1}{v}=-\dfrac{1}{\sqrt{2\mu}}\cdot\dfrac{\sqrt{hx}}{\sqrt{(h-x)}}$

$$=-\sqrt{\frac{h}{2\mu}}\cdot\frac{x}{\sqrt{(hx-x^2)}}.$$

To integrate, $-\dfrac{x}{\sqrt{(hx-x^2)}}=\dfrac{\frac{1}{2}h-x}{\sqrt{(hx-x^2)}}-\dfrac{\frac{1}{2}h}{\sqrt{(hx-x^2)}}$

$$=\frac{\frac{1}{2}h-x}{\sqrt{(hx-x^2)}}-\frac{\frac{1}{2}h}{\sqrt{\{\frac{1}{4}h^2-(\frac{1}{2}h-x)^2\}}}.$$

$$\int_x-\frac{x}{\sqrt{(hx-x^2)}}=\sqrt{(hx-x^2)}-\frac{h}{2}\cos^{-1}.\frac{\frac{1}{2}h-x}{\frac{1}{2}h}+C.$$

And since the time begins when $x=h$, $C=\dfrac{h}{2}\,.\,\pi$,

$$t=\sqrt{\frac{h}{2\mu}}\left\{\sqrt{(hx-x^2)}+\frac{h}{2}\cos^{-1}\frac{x-\frac{1}{2}h}{\frac{1}{2}h}\right\}.$$

When x is 0 the arc is $\cos^{-1}-1$, or π, and $t=\dfrac{\pi h^{\frac{3}{2}}}{2\sqrt{2\mu}}$.

40. At the distance x from the centre of force

$$v^2=2\mu\left\{\frac{1}{x}-\frac{1}{h}\right\}.$$

But if a body revolve about the centre of force at the distance x, with a velocity u,

$$u^2 = 2\frac{\mu}{x^2} \cdot \frac{x}{2} = \frac{\mu}{x}.$$

And these are equal if

$$2\mu\left(\frac{1}{x} - \frac{1}{h}\right) = \frac{\mu}{x}; \quad \text{or if } h = 2x;$$

that is, a body must fall from a distance equal to twice the radius, in order to acquire the velocity in the circle.

41. In the ellipse, the velocity is equal to that acquired by falling with the force at P through $\frac{1}{4}$ the chord of curvature. The chord of curvature is (see the diagrams in B. III. and IV.)

$$\frac{2CD^2}{AC} = \frac{2SP \,.\, HP}{AC} = \frac{2x(2a - x)}{a}.$$

Hence $\text{velocity}^2 = \frac{2\mu}{x^2} \cdot \frac{x(2a - x)}{2a} = 2\mu\left(\frac{1}{x} - \frac{1}{2a}\right)$;

and this will be equal to v^2, the velocity acquired by falling, if $h = 2a$.

Therefore a body falling from a distance equal to the major axis of the ellipse, acquires, on reaching the curve, the velocity of a body in the curve.

THE END.